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THE
ESSENTIALS OF GEOMETRY
(SOLID)

BY

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PREFACE.

In the *Essentials of Geometry*, the author has endeavored to prepare a work suited to the needs of high schools and academies. It will also be found to answer as well the requirements of colleges and scientific schools.

In some of its features, the work is similar to the author's *Revised Plane and Solid Geometry;* but important improvements have been introduced, which are in line with the present requirements of many progressive teachers.

In a number of propositions, the figure is given, and a statement of what is to be proved; the details of the proof being left to the pupil, usually with a hint as to the method of demonstration to be employed.

The propositions and corollaries left in this way for the pupil to demonstrate, in the Plane Geometry, will be found in the following sections: —

Book I., §§ 51, 75, 76, 78, 79, 96, 102, 110, 111, 112, 115, 117, 136.

Book II., §§ 158, 160, 165, 170, 172 (Case III.), 174, 178, 179, 193 (Case III.), 194, and 201.

Book III., §§ 251, 257, 261, 264, 268, 278, 282, 284, and 286.

Book IV., §§ 312 and 316.

Book V., §§ 346, 347, and 350.

Book VI., §§ 405, 407, 412, 414, 415, 416, 417, 420, 421, 434, 437, 440, 442, and 444.

Book VII., §§ 491, 495, 507, 512, 513, 521, 528, 529, and 530.

Book VIII., §§ 554, 559, 578, 580, 581, 594, 595, 601, 603, 608, 613, 614, 625 (Case II.), 630, 631, 635, and 637.

Book IX., §§ 654, 656, 660, 673, and 679.

There are also Problems in Construction in which the construction or proof is left to the pupil.

Another important improvement consists in giving figures and suggestions for the exercises. In Book I., the pupil has a figure for every non-numerical exercise; after that, they are only given with the more difficult ones.

In many of the exercises in construction, the pupil is expected to discuss the problem, or point out its limitations.

In Book I., and also in the first eighteen propositions of Book VI., the authority for each statement of a proof is given directly after the statement, in smaller type, enclosed in brackets. In the remaining portions of the work, the formal statement of the authority is omitted; but the number of the section where it is to be found is usually given.

In a number of cases, however, where the pupil is presumed, from practice, to be so familiar with the authority as not to require reference to the section where it is to be found, there is given merely an interrogation-point.

In all these cases the pupil should be required to give the authority as carefully and accurately as if it were actually printed on the page.

Another improvement consists in marking the parts of a demonstration by the words *Given*, *To Prove*, and *Proof*, printed in heavy-faced type.

A similar system is followed in the Constructions, by the use of the words *Given*, *Required*, *Construction*, and *Proof*.

A minor improvement is the omission of the definite article in speaking of geometrical magnitudes; thus we speak of "angle *A*," "triangle *ABC*," etc., and not "the angle *A*," "the triangle *ABC*," etc.

Symbols and abbreviations have been freely used; a list of these will be found on page 4.

Particular attention has been given to putting the propositions in the first part of Book I. in a form adapted to the needs of a beginner.

The pages have been arranged in such a way as to avoid the necessity, while reading a proof, of turning the page for reference to the figure.

The Appendix to the Plane Geometry contains propositions on Maxima and Minima of Plane Figures, and Symmetrical Figures; also, additional exercises of somewhat greater difficulty than those previously given.

The Appendix to the Solid Geometry contains rigorous proofs of the limit statements made in §§ 639, 650, 667, and 674.

The author wishes to acknowledge, with thanks, the many suggestions which he has received from teachers in all parts of the country, which have added materially to the value of the work.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
1899.

Stereoscopic views of many of the figures in the Solid Geometry have been prepared. Full particulars may be obtained from the publishers.

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SOLID GEOMETRY.

Book VI.

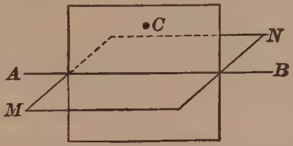
LINES AND PLANES IN SPACE. DIEDRAL ANGLES. POLYEDRAL ANGLES.

394. Def. A plane is said to be *determined* by certain lines or points when one plane, and only one, can be drawn through these lines or points.

Prop. I. Theorem.

395. *A plane is determined*

I. *By a straight line and a point without the line.*
II. *By three points not in the same straight line.*
III. *By two intersecting straight lines.*
IV. *By two parallel straight lines.*

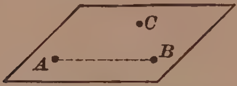


I. **Given** point C without str. line AB.

To Prove that a plane is determined by AB and C.

Proof. If any plane MN be drawn through AB, it may be revolved about AB as an axis until it contains point C.

Hence, a plane can be drawn through line AB and point C; and it is evident that but one such plane can be drawn.



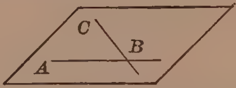
II. **Given** *A*, *B*, and *C* three points not in the same str. line.

To Prove that a plane is determined by *A*, *B*, and *C*.

Proof. Draw line *AB*; then a plane, and only one, can be drawn through line *AB* and point *C*.

[A plane is determined by a str. line and a point without the line.] (§ 395, I)

Then, a plane, and only one, can be drawn through *A*, *B*, and *C*.



III. **Given** *AB* and *BC* intersecting str. lines.

To Prove that a plane is determined by *AB* and *BC*.

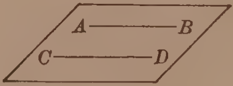
Proof. A plane, and only one, can be drawn through line *AB* and point *C*.

[A plane is determined by a str. line and a point without the line.] (§ 395, I)

And since this plane contains points *B* and *C*, it must contain line *BC*.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 9)

Then, a plane, and only one, can be drawn through *AB* and *BC*.



IV. **Given** ∥s *AB* and *CD*.

To Prove that a plane is determined by *AB* and *CD*.

Proof. The ∥s *AB* and *CD* lie in the same plane.

[Two str. lines are said to be ∥ when they lie in the same plane, and cannot meet however far they may be produced.] (§ 52)

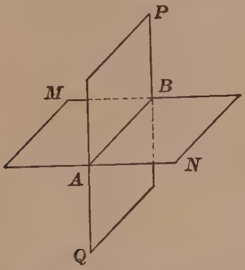
And only one plane can be drawn through AB and point C.

[A plane is determined by a str. line and a point without the line.] (§ 395, I)

Then, a plane, and only one, can be drawn through AB and CD.

PROP. II. THEOREM.

396. *The intersection of two planes is a straight line.*



Given line AB the intersection of planes MN and PQ.

To Prove AB a str. line.

Proof. Draw a str. line between points A and B.

This str. line lies in plane MN, and also in plane PQ.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 9)

Then it must be the intersection of planes MN and PQ.

Hence, the line of intersection AB is a str. line.

397. Defs. If a straight line meets a plane, the point of intersection is called the *foot* of the line.

A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line drawn in the plane through its foot.

A straight line is said to be *parallel to a plane* when it cannot meet the plane however far they may be produced.

A straight line which is neither perpendicular nor parallel to a plane, is said to be *oblique* to it.

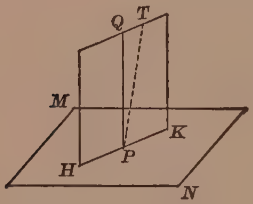
Two planes are said to be *parallel to each other* when they cannot meet however far they may be produced.

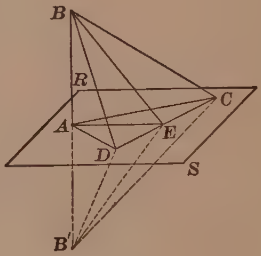
398. Sch. The following is given for convenience of reference:

A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.

PROP. III. THEOREM.

399. *At a given point in a plane, one perpendicular to the plane can be drawn, and but one.*





Given point P in plane MN.

To Prove that a $\perp$ can be drawn to MN at P, and but one.

Proof. At any point A of indefinite str. line AB, draw lines AC and $AD \perp$ to AB.

Let RS be the plane determined by AC and AD.

Let AE be any other str. line drawn through point A in plane RS; and draw line CD intersecting AC, AE, and AD at C, E, and D, respectively.

Produce BA to B', making $AB' = AB$, and draw lines BC, BE, BD, $B'C$, $B'E$, and $B'D$.

In $\triangle$s BCD and $B'CD$,

$$CD = CD.$$

And since AC and AD are $\perp$ to BB' at its middle point,

$$BC = B'C \text{ and } BD = B'D.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41, I)

$$\therefore \triangle BCD = \triangle B'CD.$$

[Two $\triangle$ are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69)

Now revolve $\triangle BCD$ about CD as an axis until it coincides with $\triangle B'CD$.

Then, point B will fall at point B', and line BE will coincide with line $B'E$; that is, $BE = B'E$.

Hence, since points A and E are each equally distant from B and B', line AE is $\perp BB'$.

[Two points, each equally distant from the extremities of a str. line, determine a $\perp$ at its middle point.] (§ 43)

But AE is *any* str. line drawn through A in plane RS.

Then, AB is $\perp$ to *every* str. line drawn through A in plane RS.

Whence, AB is $\perp$ to plane RS.

[A str. line is said to be $\perp$ to a plane when it is $\perp$ to every str. line drawn in the plane through its foot.] (§ 397)

Now apply plane RS to plane MN so that point A shall fall at point P; and let AB take the position PQ.

Then, PQ will be $\perp MN$.

Hence, a $\perp$ can be drawn to MN at P.

If possible, let PT be another $\perp$ to plane MN at P; and let the plane determined by PQ and PT intersect MN in line HK.

Then, both PQ and PT are $\perp HK$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

But this is impossible; for, in plane HKT, only one $\perp$ can be drawn to HK at P.

[At a given point in a str. line, but one $\perp$ to the line can be drawn.] (§ 25)

Then only one $\perp$ can be drawn to MN at P.

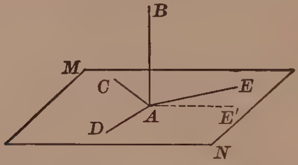
400. Cor. I. *A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.*

401. Cor. II. Since E is any point in plane RS, it follows that

If a plane is perpendicular to a straight line at its middle point, any point in the plane is equally distant from the extremities of the line.

PROP. IV. THEOREM.

402. *All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.*



Given AC, AD, and AE any three ⊥s to line AB at A.

To Prove that they lie in a plane ⊥ to AB.

Proof. Let MN be the plane determined by AC and AD. Then, plane MN is ⊥ AB.

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

Let the plane determined by AB and AE intersect MN in line AE'; then, $AB \perp AE'$.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

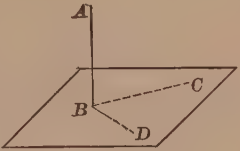
But in plane ABE, only one ⊥ can be drawn to AB at A.

[At a given point in a str. line, but one ⊥ to the line can be drawn.] (§ 25)

Then, AE' coincides with AE, and AE lies in plane MN.
But AC, AD, and AE are any three ⊥s to AB at A.
Therefore, all the ⊥s to AB at A lie in a plane ⊥ AB.

403. Cor. I. *Through a given point in a straight line, a plane can be drawn perpendicular to the line, and but one.*

404. Cor. II. *Through a given point without a straight line, a plane can be drawn perpendicular to the line, and but one.*



Given point C without line AB.

To Prove that a plane can be drawn through $C \perp AB$, and but one.

Proof. Draw line $CB \perp AB$, and let BD be any other $\perp$ to AB at B.

Then, the plane determined by BC and BD will be a plane drawn through $C \perp AB$.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 400)

Again, every plane through $C \perp AB$ must intersect the plane determined by AB and BC in a line from $C \perp AB$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

But only one $\perp$ can be drawn from C to AB.

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 45)

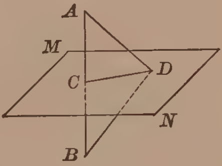
Then, every plane through $C \perp AB$ must contain BC, and be $\perp$ to AB at B.

But only one plane can be drawn through $B \perp AB$.

[Through a given point in a str. line, but one plane can be drawn $\perp$ to the line.] (§ 403)

Hence, but one plane can be drawn through $C \perp AB$.

405. Cor. III. (Converse of § 401.) *Any point equally distant from the extremities of a straight line lies in a plane perpendicular to the line at its middle point.*



Given plane $MN \perp$ to line AB at its middle point C, and point D equally distant from A and B.

To Prove that D lies in MN.

(By § 43, $CD \perp AB$; then use § 402.)

Note. It follows from §§ 401 and 405 that

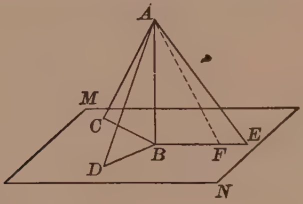
The locus (§ 141) of points in space equally distant from the extremities of a straight line is a plane perpendicular to the line at its middle point.

PROP. V. THEOREM.

406. *If from a point in a perpendicular to a plane, oblique lines be drawn to the plane,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



I. **Given** line $AB \perp$ to plane MN at B, and AC and AD oblique lines meeting MN at equal distances from B.

To Prove $AC = AD$.

Proof. Draw lines BC and BD.

In $\triangle ABC$ and ABD, $AB = AB$.

Also, $\angle ABC = \angle ABD$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

And by hyp., $BC = BD$.

$\therefore \triangle ABC = \triangle ABD$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 63)

$\therefore AC = AD$.

[In equal figures, the homologous parts are equal.] (§ 66)

II. **Given** line $AB \perp$ to plane MN at B, and AC and AE oblique lines from A to MN, AE meeting MN at a greater distance from B than AC.

To Prove $AE > AC$.

Proof. Draw lines BC and BE.

On BE take $BF = BC$, and draw line AF.

Since AF and AC meet MN at equal distances from B,

$$AF = AC.$$

[If from a point in a ⊥ to a plane, oblique lines be drawn to the plane, two oblique lines cutting off equal distances from the foot of the ⊥ are equal.] (§ 406, I)

But, $AB \perp BE$.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

$$\therefore AE > AF.$$

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the ⊥ from the point to the line, the more remote is the greater.] (§ 49, II)

$$\therefore AE > AC.$$

PROP. VI. THEOREM.

407. (Converse of Prop. V.) *If from a point in a perpendicular to a plane, oblique lines be drawn to the plane,*

I. *Two equal oblique lines cut off equal distances from the foot of the perpendicular.*

II. *Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular.*

I. **Given** line $AB \perp$ to plane MN at B, AC and AD equal oblique lines from A to MN, and lines BC and BD. (Fig. of Prop. V.)

To Prove $BC = BD$.

(Prove △ ABC and ABD equal.)

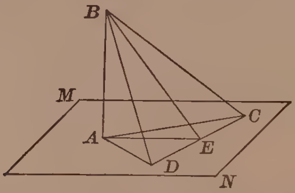
II. **Given** line $AB \perp$ to plane MN at B, and AC and AE oblique lines from A to MN, AE being $> AC$; also, lines BC and BE.

To Prove $BE > BC$.

(The proof is left to the pupil.)

PROP. VII. THEOREM.

408. *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Given line $AB \perp$ to plane MN at A, line $AE \perp$ to any line CD in MN, and line BE from E to any point B in AB.

To Prove $BE \perp CD$.

Proof. On CD take $EC = ED$.

Draw lines AC, AD, BC, and BD.

$$\therefore AC = AD.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41, I)

$$\therefore BC = BD.$$

[If from a point in a $\perp$ to a plane, oblique lines be drawn to the plane, two oblique lines cutting off equal distances from the foot of the $\perp$ are equal.] (§ 406, I)

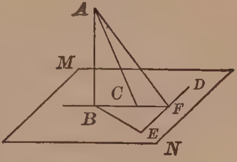
Then since each of the points B and E is equally distant from C and D,

$$BE \perp CD.$$

[Two points, each equally distant from the extremities of a str. line, determine a $\perp$ at its middle point.] (§ 43)

409. Cor. I. *From a given point without a plane, one perpendicular to the plane can be drawn, and but one.*

Given point A without plane MN.



To Prove that a $\perp$ can be drawn from A to MN, and but one.

Proof. Let DE be any line in plane MN; draw line $AF \perp DE$, line BF in plane $MN \perp DE$, line $AB \perp BF$, and line BE.

Now EF is $\perp$ to the plane determined by AF and BF.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 400)

Then since BF is drawn through the foot of EF, $\perp$ to line AB in plane ABF, we have $BE \perp AB$.

[If through the foot of a $\perp$ to a plane a line be drawn at rt. $\angle$s to any line in the plane, the line drawn from its intersection with this line to any point in the $\perp$ will be $\perp$ to the line in the plane.] (§ 408)

Then AB, being $\perp$ to BE and BF, is $\perp$ to MN.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 400)

If possible, let AC be another $\perp$ from A to MN; then $\triangle ABC$ will have two rt. $\angle$s.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

But this is impossible.

Hence, but one $\perp$ can be drawn from A to MN.

410. Cor. II. *The perpendicular is the shortest line that can be drawn from a point to a plane.*

Given AB the $\perp$ from point A to plane MN, and AC any other str. line from A to MN. (Fig. of § 409.)

To Prove $AB < AC$.

Proof. Draw line BC; then, $AB \perp BC$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

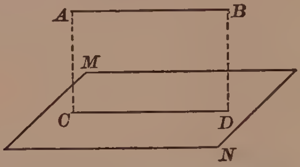
$$\therefore AB < AC.$$

[The $\perp$ is the shortest line that can be drawn from a point to a str. line.] (§ 46)

Note. The *distance* of a point from a plane signifies the length of the perpendicular from the point to the plane.

PROP. VIII. THEOREM.

411. *If two straight lines are parallel, a plane drawn through one of them, not coinciding with the plane of the parallels, is parallel to the other.*



Given line *AB* ‖ to line *CD*, and plane *MN* drawn through *CD*, not coinciding with the plane of the ‖s.

To Prove *AB* ‖ *MN*.

Proof. The ‖s *AB* and *CD* lie in a plane which intersects *MN* in line *CD*.

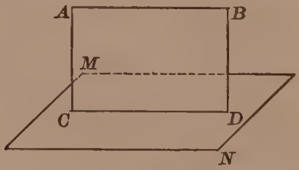
Hence, if *AB* meets *MN*, it must be at some point of *CD*.

But since *AB* is ‖ *CD*, it cannot meet *CD* (§ 52).

Then *AB* and *MN* cannot meet, and are ‖ (§ 397).

PROP. IX. THEOREM.

412. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Given line *AB* ‖ to plane *MN*; and line *CD* the intersection of *MN* with any plane *AD* drawn through *AB*.

To Prove *AB* ‖ *CD*.

(*AB* and *CD* lie in the same plane, and cannot meet.)

413. Cor. *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*

Given line $AB \parallel$ to plane MN; and line CD through any point C of $MN \parallel$ to AB. (Fig. of Prop. IX.)

To Prove that CD lies in MN.

Proof. The plane determined by line AB and point C intersects MN in a line $\parallel$ to AB.

[If a str. line is $\parallel$ to a plane, the intersection of the plane with any plane drawn through the line is $\parallel$ to the line.] (§ 412)

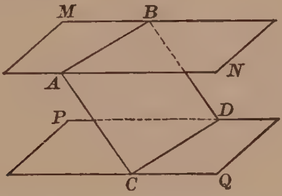
But through C, only one $\parallel$ can be drawn to AB.

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.] (§ 53)

Whence, CD lies in MN.

Prop. X. Theorem.

414. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Given $\parallel$ planes MN and PQ cut by plane AD in lines AB and CD, respectively.

To Prove $AB \parallel CD$.

(AB and CD lie in the same plane, and cannot meet.)

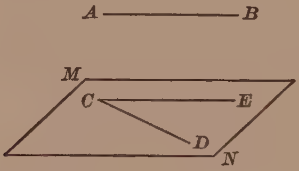
415. Cor. *Parallel lines included between parallel planes are equal.*

Given AC and $BD \parallel$ lines included between $\parallel$ planes MN and PQ. (Fig. of Prop. X.)

(Prove $AC = BD$ by §§ 414 and 107.)

PROP. XI. THEOREM.

416. *Through any given straight line, a plane can be drawn parallel to any other straight line.*



Given lines AB and CD.

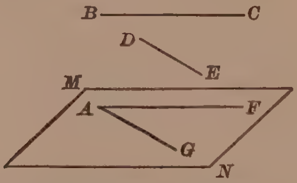
To Prove that a plane can be drawn through $CD \parallel AB$.

(Draw line $CE \parallel AB$; then use § 411.)

Note. If AB is $\parallel$ CD, an indefinitely great number of planes can be drawn through $CD \parallel AB$ (§ 411); otherwise, but one such plane can be drawn, for every plane drawn through $CD \parallel AB$ must contain CE (§ 413), and but one plane can be drawn through CD and CE.

PROP. XII. THEOREM.

417. *Through a given point a plane can be drawn parallel to any two straight lines in space.*



Given point A and lines BC and DE.

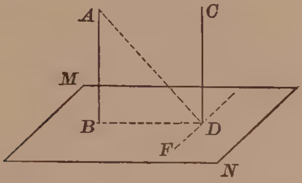
To Prove that a plane can be drawn through $A \parallel$ to BC and DE.

(The proof is left to the pupil; see § 411.)

Note. If BC and DE are $\parallel$, an indefinitely great number of planes can be drawn through $A \parallel$ to BC and DE (§ 411); otherwise, but one such plane can be drawn.

PROP. XIII. THEOREM.

418. *Two perpendiculars to the same plane are parallel.*



Given lines AB and $CD \perp$ to plane MN at B and D, respectively.

To Prove $AB \parallel CD$.

Proof. Let A be any point of AB, and draw line AD. Also, draw line BD, and line DF in plane $MN \perp BD$.

$$\therefore CD \perp DF.$$

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

Also, $AD \perp DF$.

[If through the foot of a ⊥ to a plane a line be drawn at rt. ⦞ to any line in the plane, the line drawn from its intersection with this line to any point in the ⊥ will be ⊥ to the line in the plane.] (§ 408)

Then, CD, AD, and BD, being ⊥ to DF at D, lie in the same plane.

[All the ⊥s to a str. line at a given point lie in a plane ⊥ to the line.] (§ 402)

Then, since points A and B lie in the plane of the lines AD, BD, and CD, AB lies in this plane.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 9)

That is, AB and CD lie in the same plane.

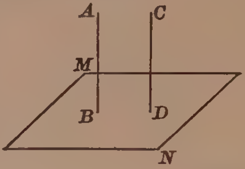
Again, AB and CD are $\perp BD$.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

$$\therefore AB \parallel CD.$$

[Two ⊥s to the same str. line are ∥.] (§ 54)

419. Cor. I. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Given lines AB and CD ∥, and $AB \perp$ to plane MN.

To Prove $CD \perp MN$.

Proof. A ⊥ from C to MN will be ∥ AB.

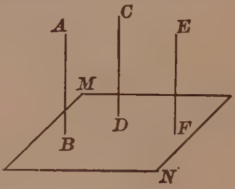
[Two ⊥s to the same plane are ∥.] (§ 418)

But through C, only one ∥ can be drawn to AB.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

$$\therefore CD \perp MN.$$

420. Cor. II. *If each of two straight lines is parallel to a third straight line, they are parallel to each other.*



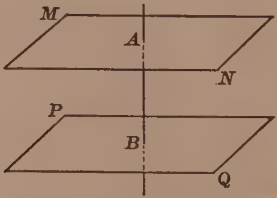
Given lines AB and CD ∥ line EF.

To Prove $AB \parallel CD$.

(Draw plane $MN \perp EF$, and prove $AB \parallel CD$ by §§ 418 and 419.)

Prop. XIV. Theorem.

421. *Two planes perpendicular to the same straight line are parallel.*



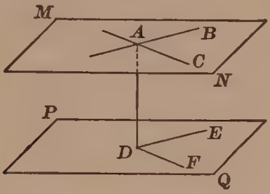
Given planes MN and $PQ \perp$ to line AB.

To Prove $MN \parallel PQ$.

(Prove as in § 54; by § 404, but one plane can be drawn through a given point ⊥ to a given str. line.)

PROP. XV. THEOREM.

422. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Given lines AB and AC, in plane MN, ∥ to plane PQ.

To Prove $MN \parallel PQ$.

Proof. Draw line $AD \perp PQ$, and lines DE and DF ∥ to AB and AC, respectively; then DE and DF lie in plane PQ.

[If a line and a plane are ∥, a ∥ to the line through any point of the plane lies in the plane.] (§ 413)

Whence, AD is ⊥ to DE and DF.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

Therefore, AD is ⊥ to AB and AC.

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

$$\therefore AD \perp MN.$$

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

$$\therefore MN \parallel PQ.$$

[Two planes ⊥ to the same str. line are ∥.] (§ 421)

EXERCISES.

1. What is the locus (§ 141) of the perpendiculars to a given straight line at a given point?

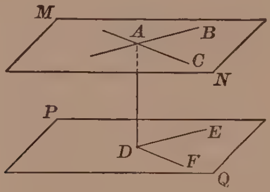
2. What is the locus of points in space equally distant from the circumference of a given circle?

3. A line parallel to a plane is everywhere equally distant from it.

(Fig. of Prop. IX. Draw lines AC and $BD \perp MN$. To prove $AC = BD$.)

Prop. XVI. Theorem.

423. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given MN and PQ ∥ planes, and line $AD \perp PQ$.

To Prove $AD \perp MN$.

Proof. Pass two planes through AD, intersecting MN in lines AB and AC, and PQ in lines DE and DF, respectively.

Then, $AB \parallel DE$, and $AC \parallel DF$.

[If two ∥ planes are cut by a third plane, the intersections are ∥.] (§ 414)

But AD is ⊥ to DE and DF.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

Whence, AD is ⊥ to AB and AC.

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

$$\therefore AD \perp MN.$$

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

424. Cor. I. *Two parallel planes are everywhere equally distant.* (Note, p. 244.)

Given MN and PQ ∥ planes. (Fig. of Prop. XVI.)

To Prove MN and PQ everywhere equally distant.

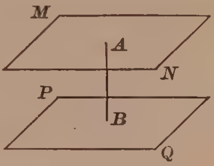
Proof. All lines which are ⊥ to both planes are ∥.

[Two ⊥s to the same plane are ∥.] (§ 418)

Therefore, these lines are all equal.

[∥ lines included between ∥ planes are equal.] (§ 415)

425. Cor. II. *Through a given point a plane can be drawn parallel to a given plane, and but one.*



Given point A and plane PQ.

To Prove that a plane can be drawn through $A \parallel PQ$, and but one.

Proof. Draw line $AB \perp PQ$.

Through A pass plane $MN \perp AB$.

Then MN will be $\parallel PQ$.

[Two planes $\perp$ to the same str. line are $\parallel$.] (§ 421)

If another plane could be drawn through $A \parallel PQ$, it would be $\perp AB$.

[A str. line $\perp$ to one of two $\parallel$ planes is $\perp$ to the other also.] (§ 423)

It would then coincide with MN.

[Through a given point in a str. line, but one plane can be drawn $\perp$ to the line.] (§ 403)

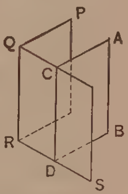
Then but one plane can be drawn through $A \parallel PQ$.

EXERCISES.

4. What is the locus of points in space equally distant from the vertices of a given triangle?

5. What is the locus of points in space equally distant from a given plane?

6. What is the locus of points in space equally distant from two parallel planes?



7. A line parallel to each of two intersecting planes is parallel to their intersection.

(Pass a plane through $AB \parallel PR$; then use § 412.)

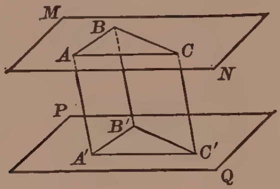
8. If two planes are parallel to a third plane, they are parallel to each other. (§§ 423, 421.)

9. Line AB is perpendicular to plane MN at A. A line is drawn from A meeting any line CD of plane MN at E. If line BE is perpendicular to CD, prove AE perpendicular to CD.

(Fig. of Prop. VII.)

PROP. XVII. THEOREM.

426. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Given $\angle$s BAC and $B'A'C'$ in planes MN and PQ, respectively, with AB and $AC \parallel$ respectively to $A'B'$ and $A'C'$, and extending in the same direction.

To Prove $\angle BAC = \angle B'A'C'$, and $MN \parallel PQ$.

Proof. Lay off $AB = A'B'$ and $AC = A'C'$, and draw lines AA', BB', CC', BC, and $B'C'$.

Then since AB is equal and $\parallel$ to $A'B'$, $ABB'A'$ is a ▱.

[If two sides of a quadrilateral are equal and $\parallel$, the figure is a ▱.] (§ 110)

Whence, AA' is equal and $\parallel$ to BB'.

[The opposite sides of a ▱ are equal.] (§ 106, I)

Similarly, $ACC'A'$ is a ▱, and AA' is equal and $\parallel$ to CC'.

Then, BB' is equal and $\parallel$ to CC'.

[If each of two str. lines is $\parallel$ to a third str. line, they are $\parallel$ to each other.] (§ 420)

Whence, $BB'C'C$ is a ▱, and $BC = B'C'$.

$$\therefore \triangle ABC = \triangle A'B'C'.$$

[Two $\triangle$s are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69)

$$\therefore \angle BAC = \angle B'A'C'.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Again, lines AB and AC are $\parallel$ to plane PQ.

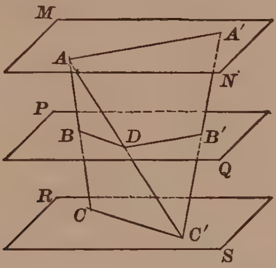
[If two str. lines are $\parallel$, a plane drawn through one of them, not coinciding with the plane of the $\parallel$s, is $\parallel$ to the other.] (§ 411)

$$\therefore MN \parallel PQ.$$

[If each of two intersecting lines is ‖ to a plane, their plane is ‖ to the given plane.] (§ 422)

PROP. XVIII. THEOREM.

427. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*



Given ‖ planes MN, PQ, and RS intersecting lines AC and $A'C'$ in points A, B, C, and A', B', C', respectively.

To Prove $$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

Proof. Draw line AC'; and through AC and AC' pass a plane intersecting PQ and RS in lines BD and CC', respectively.

$$\therefore BD \parallel CC'.$$

[If two ‖ planes are cut by a third plane, the intersections are ‖.] (§ 414)

$$\therefore \frac{AB}{BC} = \frac{AD}{DC'}. \qquad (1)$$

[A ‖ to one side of a △ divides the other two sides proportionally.] (§ 244)

In like manner, $$\frac{AD}{DC'} = \frac{A'B'}{B'C'}. \qquad (2)$$

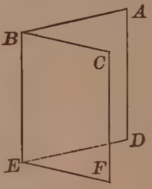
From (1) and (2), $$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

DIEDRAL ANGLES.

DEFINITIONS.

428. A *diedral angle* is the *amount of divergence* of two planes which meet in a straight line.

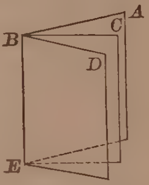


The line of intersection of the planes is called the *edge* of the diedral angle, and the planes are called its *faces*.

Thus, in the diedral angle between planes BD and BF, BE is the edge, and BD and BF the faces.

A diedral angle may be designated by two letters on its edge; or, if several diedral angles have a common edge, by four letters, one in each face and two on the edge, the letters on the edge being named between the other two.

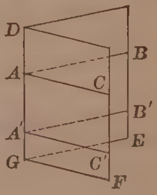
Thus, we may read the above diedral angle BE, or $ABEC$.



Two diedral angles are said to be *adjacent* when they have the same edge, and a common face between them; as, $ABEC$ and $CBED$.

Two diedral angles are said to be *vertical* when the faces of one are the extensions of the faces of the other.

429. A *plane angle* of a diedral angle is the angle between two straight lines drawn one in each face, perpendicular to the edge at the same point.



Thus, if lines AB and AC be drawn in faces DE and DF, respectively, of diedral angle DG, perpendicular to DG at A, $\angle BAC$ is a plane angle of the diedral angle.

430. Let BAC and $B'A'C'$ (Fig. of § 429) be plane $\angle$s of diedral $\angle DG$; then, $AB \parallel A'B'$ and $AC \parallel A'C'$. (§ 54)

$$\therefore \angle BAC = \angle B'A'C'. \qquad (§ 426)$$

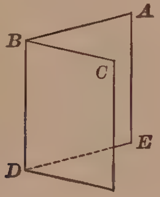
That is, *all plane angles of a diedral angle are equal.*

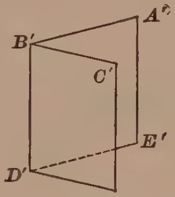
431. A plane perpendicular to the edge of a diedral angle intersects the faces in lines perpendicular to the edge (§ 398); hence, *a plane perpendicular to the edge of a diedral angle intersects the faces in lines which include the plane angle of the diedral angle* (§ 429).

432. Two diedral angles are *equal* when their faces may be made to coincide.

PROP. XIX. THEOREM.

433. *Two diedral angles are equal if their plane angles are equal.*





Given ABC and $A'B'C'$ plane $\angle$s of diedral $\angle$s BD and $B'D'$, respectively, and $\angle ABC = \angle A'B'C'$.

To Prove diedral $\angle BD =$ diedral $\angle B'D'$.

Proof. Apply diedral $\angle B'D'$ to BD in such a way that $A'B'$ shall coincide with AB, and $B'C'$ with BC.

Now BD and $B'D'$ are $\perp$ to the planes of $\angle$s ABC and $A'B'C'$, respectively. (§ 400)

Whence, $B'D'$ will coincide with BD. (§ 399)

Then, $A'D'$ will coincide with AD, and $C'D'$ with CD. (§ 395, III)

Hence, $B'D'$ and BD are equal. (§ 432)

434. Cor. I. (Converse of Prop. XIX.) *If two diedral angles are equal, their plane angles are equal.* (Fig. of Prop. XIX.)

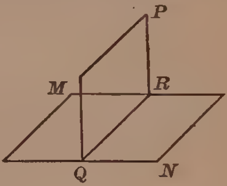
(Apply $B'D'$ to BD so that face $A'D'$ shall coincide with AD, and $C'D'$ with CD, point B' falling at B.)

435. Cor. II. *If two planes intersect, the vertical diedral angles are equal.*

For their plane angles are equal. (§ 40)

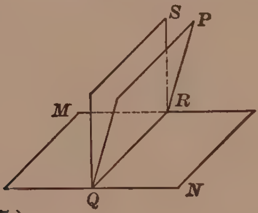
436. Defs. If a plane meets another plane in such a way as to make the adjacent diedral angles equal, each is called a *right diedral angle*, and the planes are said to be *perpendicular to each other.*

Thus, if plane *PQ* be drawn meeting plane *MN* in such a way as to make diedral ∠s *PRQM* and *PRQN* equal, each of these is a right diedral ∠, and *MN* and *PQ* are ⊥ to each other.



PROP. XX. THEOREM.

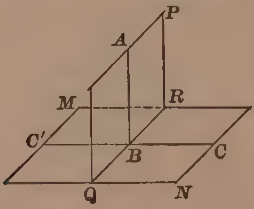
437. *Through a given line in a plane, a plane can be drawn perpendicular to the given plane, and but one.*



(Prove as in § 25.)

PROP. XXI. THEOREM.

438. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given planes PQ and $MN \perp$, intersecting in line QR, and line AB in plane $PQ \perp QR$.

To Prove $AB \perp MN$.

Proof. Draw line $C'BC$ in plane $MN \perp QR$.

Then, ABC and ABC' are plane $\angle$s of diedral $\angle$s $PRQN$ and $PRQM$, respectively. (§ 429)

Now, if two planes are $\perp$ to each other, the adj. diedral $\angle$s are equal (§ 436).

That is, diedral $\angle PRQN =$ diedral $\angle PRQM$.

$$\therefore \angle ABC = \angle ABC'. \qquad (§\ 434)$$

Whence, $\angle ABC$ is a rt. $\angle$. (§ 24)

Then AB, being $\perp$ to BC and BQ at B, is $\perp MN$. (§ 400)

439. Cor. I. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

Given planes PQ and $MN \perp$, intersecting in line QR, and line AB drawn from any point B of $QR \perp MN$. (Fig. of Prop. XXI.)

To Prove that AB lies in PQ.

Proof. If a line be drawn in PQ from point $B \perp QR$, it will be $\perp MN$. (§ 438)

But from point B but one $\perp$ can be drawn to MN. (§ 399)

Therefore, AB lies in PQ.

440. Cor. II. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other lies in the other.*

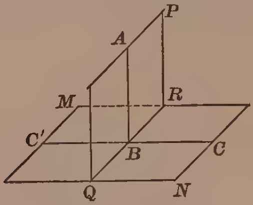
Given planes PQ and $MN \perp$, intersecting in line QR, and line AB drawn from any point A of $PQ \perp MN$. (Fig. of Prop. XXI.)

To Prove that AB lies in PQ.

(The proof is left to the pupil.)

PROP. XXII. THEOREM.

441. *If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.*



Given line $AB \perp$ plane MN, and PQ any plane drawn through AB.

To Prove $PQ \perp MN$.

Proof. Let line QR be the intersection of PQ and MN, and draw line $C'BC$ in plane $MN \perp QR$.

We have $AB \perp BQ$. (§ 398)

Then, ∡ ABC and ABC' are plane ∡ of diedral ∡ $PRQN$ and $PRQM$, respectively. (§ 429)

But ∡ ABC and ABC' are rt. ∡. (§ 398)

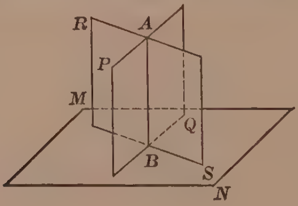
$\therefore \angle ABC = \angle ABC'$. (§ 26)

$\therefore$ diedral $\angle PRQN =$ diedral $\angle PRQM$. (§ 433)

$\therefore PQ \perp MN$. (§ 436)

PROP. XXIII. THEOREM.

442. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



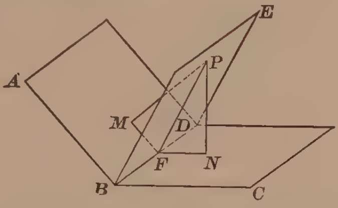
Given planes PQ and $RS \perp$ to plane MN, and intersecting in line AB.

To Prove $AB \perp MN$.

(By § 439, a $\perp$ to MN at B lies in both PQ and RS.)

PROP. XXIV. THEOREM.

443. *Every point in the bisecting plane of a diedral angle is equally distant from its faces.*



Given P any point in bisecting plane BE of diedral $\angle ABDC$, and lines PM and $PN \perp$ to AD and CD, respectively.

To Prove $PM = PN$.

Proof. Let the plane determined by PM and PN intersect planes AD, BE, and CD in lines FM, FP, and FN, respectively.

Plane $PMFN$ is $\perp$ to planes AD and CD. (§ 441)

Then, plane $PMFN$ is $\perp BD$. (§ 442)

Whence, PFM and PFN are plane $\angle$s of diedral $\angle$s $ABDE$ and $CBDE$, respectively. (§ 431)

$$\therefore \angle PFM = \angle PFN. \quad (§ 434)$$

In $\triangle$s PFM and PFN, $PF = PF$.

And, $\angle PFM = \angle PFN$.

Also, $\angle$s PMF and PNF are rt. $\angle$s. (§ 398)

$$\therefore \triangle PFM = \triangle PFN. \quad (§ 70)$$

$$\therefore PM = PN. \quad (?)$$

444. Cor. I. (Converse of Prop. XXIV.) *Any point which is within a diedral angle, and equally distant from its faces, lies in the bisecting plane of the diedral angle.*

Given point P within diedral $\angle ABDC$, equally distant from AD and CD, and plane BE determined by BD and P. (Fig. of Prop. XXIV.)

To Prove that BE bisects diedral $\angle ABDC$.

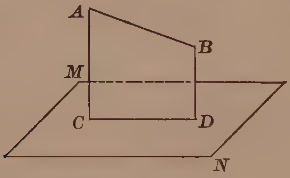
(Prove $\triangle$ PFM and PFN equal; then $\angle PFM = \angle PFN$, and the theorem follows by § 433.)

445. Cor. II. It follows from §§ 443 and 444 that

The locus of points in space equally distant from the faces of a diedral angle is the plane bisecting the diedral angle.

PROP. XXV. THEOREM.

446. *Through a given straight line without a plane, a plane can be drawn perpendicular to the given plane, and but one.*



Given line AB without plane MN.

To Prove that a plane can be drawn through $AB \perp MN$, and but one.

Proof. Draw line $AC \perp MN$, and let AD be the plane determined by AB and AC; then, $AD \perp MN$. (§ 441)

If more than one plane could be drawn through $AB \perp MN$, their common intersection, AB, would be $\perp MN$. (§ 442)

Hence, but one plane can be drawn through $AB \perp MN$, unless AB is $\perp MN$.

Note. If line AB is $\perp MN$, an indefinitely great number of planes can be drawn through $AB \perp MN$ (§ 441).

447. Defs. *The projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line on a plane* is the line which contains the projections of all its points.

448. Cor. *The projection of a straight line on a plane is a straight line.*

Given line CD the projection (§ 447) of str. line AB on plane MN. (Fig. of Prop. XXV.)

To Prove CD a str. line.

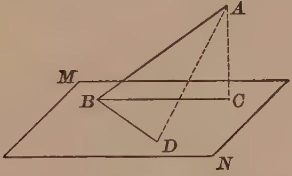
Proof. Draw a plane through $AB \perp MN$.

The $\perp$s to MN from all points of AB will lie in this plane. (§ 440)

Therefore, CD is a str. line. (§ 396)

PROP. XXVI. THEOREM.

449. *The angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*



Given line BC the projection of line AB on plane MN, and BD any other line drawn through B in MN.

To Prove $\angle ABC < \angle ABD$.

Proof. Lay off $BD = BC$, and draw lines AC and AD.

In $\triangle$s ABC and ABD, $AB = AB$.

And by hyp., $BC = BD$.

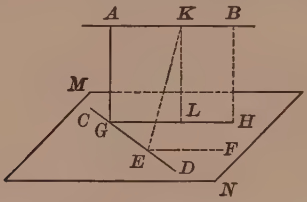
Also, $AC < AD$. (§ 410)

$\therefore \angle ABC < \angle ABD$. (§ 92)

Note. $\angle ABC$ is called the *angle* between line AB and plane MN.

PROP. XXVII. THEOREM.

450. *Two straight lines, not in the same plane, have one common perpendicular, and but one; and this line is the shortest line that can be drawn between them.*



Given lines AB and CD, not in the same plane.

To Prove that one common $\perp$ to AB and CD can be drawn, and but one; and that this line is the shortest line that can be drawn between AB and CD.

Proof. Through CD draw plane $MN \parallel AB$. (§ 416)

Through AB draw plane $AH \perp MN$, and produce their intersection to meet CD at G. (§ 446)

Draw line AG in plane $AH \perp GH$; then, $AG \perp MN$. (§ 438)

$\therefore AG \perp CD$. (§ 398)

Also, $GH \parallel AB$. (§ 412)

$\therefore AG \perp AB$. (§ 56)

Then, AG is a common $\perp$ to AB and CD.

If possible, let EK be another common $\perp$ to AB and CD, and draw line $EF \parallel AB$, and line KL in plane $AH \perp GH$.

Then, EF lies in plane MN. (§ 413)

Also, EK is $\perp$ to ED and EF. (§ 56)

Whence, EK is $\perp$ MN. (?)

But KL is also $\perp$ MN. (§ 438)

We should then have two $\perp$s from K to MN, which is impossible. (§ 409)

Hence, but one common $\perp$ can be drawn to AB and CD.

Again, $EK > KL$. (§ 410)

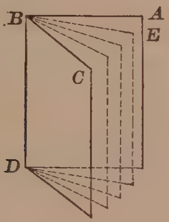
$$\therefore\ EK > AG. \qquad (\S\ 80)$$

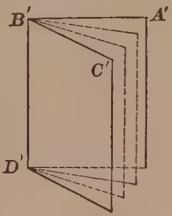
Hence, AG is the shortest line between AB and CD.

PROP. XXVIII. THEOREM.

451. *Two diedral angles are to each other as their plane angles.*

Case I. *When the plane angles are commensurable.*





Given ABC and $A'B'C'$, plane $\angle$s of diedral $\angle$s $ABDC$ and $A'B'D'C'$, respectively, and commensurable.

To Prove $$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}.$$

Proof. Let $\angle ABE$ be a common measure of $\angle$s ABC and $A'B'C'$; and suppose it to be contained 4 times in $\angle ABC$ and 3 times in $\angle A'B'C'$.

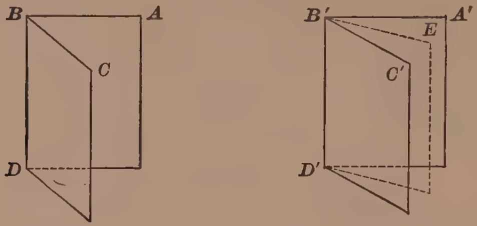
$$\therefore\ \frac{\angle ABC}{\angle A'B'C'} = \frac{4}{3}. \qquad (1)$$

Passing planes through edges BD and $B'D'$, and the several lines of division of $\angle$s ABC and $A'B'C'$, respectively, diedral $\angle ABDC$ will be divided into 4 parts, and diedral $\angle A'B'D'C'$ into 3 parts, all of which parts are equal. (§ 433)

$$\therefore\ \frac{ABDC}{A'B'D'C'} = \frac{4}{3}. \qquad (2)$$

From (1) and (2), $$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}. \qquad (?)$$

Case II. *When the plane angles are incommensurable.*



Given ABC and $A'B'C'$ plane $\angle$s of diedral $\angle$s $ABDC$ and $A'B'D'C'$, respectively, and incommensurable.

To Prove $$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}.$$

Proof. Let $\angle ABC$ be divided into any number of equal parts, and let one of these parts be applied to $\angle A'B'C'$ as a unit of measure.

Since $\angle$s ABC and $A'B'C'$ are incommensurable, a certain number of the parts will extend from $A'B'$ to $B'E$, leaving a remainder $\angle EB'C' <$ one of the equal parts.

Pass a plane through $B'D'$ and $B'E$; then since the plane $\angle$s of diedral $\angle$s $A'B'D'E$ and $ABDC$ are commensurable,

$$\frac{ABDC}{A'B'D'E} = \frac{\angle ABC}{\angle A'B'E}. \qquad \text{(§ 451, Case I)}$$

Now let the number of subdivisions of $\angle ABC$ be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder $\angle EB'C'$ will approach the limit 0.

Then $\dfrac{ABDC}{A'B'D'E}$ will approach the limit $\dfrac{ABDC}{A'B'D'C'}$,

and $\dfrac{\angle ABC}{\angle A'B'E}$ will approach the limit $\dfrac{\angle ABC}{\angle A'B'C'}$.

By the Theorem of Limits, these limits are equal. (§ 188)

$$\therefore \frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}.$$

Note. It follows from § 451 that the plane angle may be taken as the *measure* of the diedral angle; thus, if the plane angle contains n degrees, the diedral angle may be regarded as being of n degrees.

EXERCISES.

10. A straight line and a plane perpendicular to the same straight line are parallel.

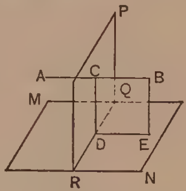
(Fig. of Prop. IX. Let plane determined by AB and AC intersect MN in CD.)

11. If two planes are parallel, a line parallel to one of them through any point of the other lies in the other.

(Fig. of Prop. X. Given planes MN and $PQ \parallel$, and AB through any point A of $MN \parallel PQ$. Prove that AB lies in MN by § 413.)

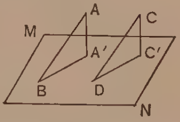
12. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

(Draw line $CD \perp QR$, and prove it $\perp MN$.)



13. If two parallels meet a plane, they make equal angles with it.

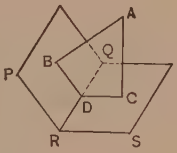
(Given $AB \parallel CD$; to prove $\angle ABA' = \angle CDC'$.)



14. If a straight line intersects two parallel planes, it makes equal angles with them.

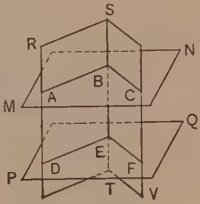
15. The angle between perpendiculars to the faces of a diedral angle from any point within the angle is the supplement of its plane angle.

(Prove $\angle BDC$ the plane $\angle$ of diedral $\angle PQRS$.)



16. If each of two intersecting planes be cut by two parallel planes, not parallel to their intersection, their intersections with the parallel planes include equal angles.

(To prove $\angle ABC = \angle DEF$.)

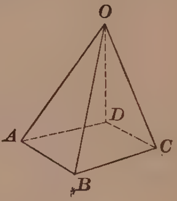


POLYEDRAL ANGLES.

DEFINITIONS.

452. A *polyedral angle* is a figure composed of three or more triangles, called *faces*, having for their bases the sides of a polygon, and for their common vertex a point without its plane; as O–$ABCD$.

The common vertex, O, is called the *vertex* of the polyedral angle, and the polygon, $ABCD$, the *base;* the vertical angles of the triangles, AOB, BOC, etc., are called the *face angles*, and their sides, OA, OB, etc., the *edges*.



Note. The polyedral angle is not regarded as limited by the base; thus, the face AOB is understood to mean the indefinite plane between the edges OA and OB produced indefinitely.

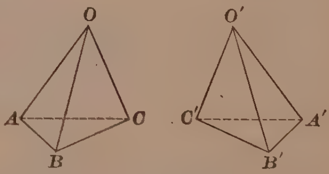
A *triedral angle* is a polyedral angle of three faces.

Two polyedral angles are called *vertical* when the edges of one are the prolongations of the edges of the other.

453. A polyedral angle is called *convex* when its base is a convex polygon (§ 121).

454. Two polyedral angles are *equal* when they can be applied to each other so that their faces shall coincide.

455. Two polyedral angles are said to be *symmetrical* when the face and diedral angles of one are equal respectively to the homologous face and diedral angles of the other, if the equal parts occur *in the reverse order*.



Thus, if face $\angle$s AOB, BOC, and COA are equal respectively to face $\angle$s $A'O'B'$, $B'O'C'$, and $C'O'A'$, and diedral $\angle$s OA, OB, and OC to diedral $\angle$s $O'A'$, $O'B'$, and $O'C'$, triedral $\angle$s O–ABC and O'–$A'B'C'$ are symmetrical.

It is evident that, in general, two symmetrical polyedral angles cannot be placed so that their faces shall coincide.

PROP. XXIX. THEOREM.

456. *Two vertical polyedral angles are symmetrical.*

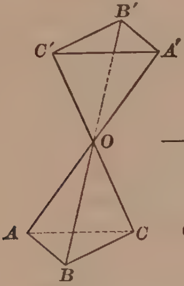


Fig. 1.

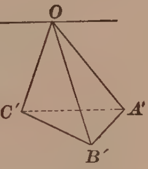


Fig. 2.

Given O–ABC and O–$A'B'C'$ (Fig. 1) vertical triedral $\angle$s.

To Prove O–ABC and O–$A'B'C'$ symmetrical.

Proof. Face $\angle$s AOB, BOC, etc., are equal, respectively, to face $\angle$s $A'OB'$, $B'OC'$, etc. (§ 40)

Again, diedral $\angle$s OA and OA' are vertical; for AOB and $A'OB'$ are portions of the same plane, as also are AOC and $A'OC'$; in like manner, diedral $\angle$s OB and OB' are vertical; etc.

Then, diedral $\angle$s OA, OB, etc., are equal, respectively, to diedral $\angle$s OA', OB', etc. (§ 435)

But the equal parts of the triedral $\angle$s occur *in the reverse order;* as may be seen by conceiving O–$A'B'C'$ moved ∥ to itself to the right, and then revolved, as shown in Fig. 2, about an axis passing through O, until face $OA'C'$ comes into the same plane as before; edge OB' being on this side of, instead of beyond, plane $OA'C'$.

Hence, O–ABC and O–$A'B'C'$ are symmetrical (§ 455).

In like manner, the theorem may be proved for any two polyedral $\angle$s.

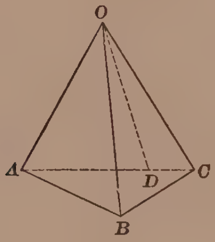
Ex. 17. If two parallel planes are cut by a third plane, the alternate-interior diedral angles are equal.

(Prove the plane ∠s of the alt.-int. diedral ∠s equal.)

PROP. XXX. THEOREM.

457. *The sum of any two face angles of a triedral angle is greater than the third.*

Note. The theorem requires proof only in the case where the third face angle is greater than either of the others.



Given in triedral $\angle O\text{–}ABC$,

face $\angle AOC >$ face $\angle AOB$ or face $\angle BOC$.

To Prove $\angle AOB + \angle BOC > \angle AOC$.

Proof. In face AOC draw line OD equal to OB, making $\angle AOD = \angle AOB$; and through B and D pass a plane cutting the faces of the triedral ∠ in lines AB, BC, and CA, respectively.

In △ AOB and AOD, $OA = OA$.

And by cons., $OB = OD$,

and $\angle AOB = \angle AOD$.

$\therefore \triangle AOB = \triangle AOD$. (?)

$\therefore AB = AD$. (?)

Now, $AB + BC > AD + DC$. (Ax. 4)

Or, since $AB = AD$, $BC > DC$.

Then, in △ BOC and COD, $OC = OC$.

Also, $OB = OD$, and $BC > DC$.

$$\therefore \angle BOC > \angle COD. \qquad (§\ 91)$$

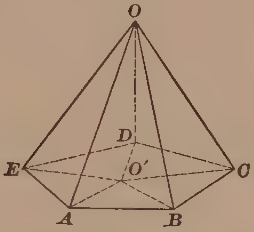
Adding $\angle AOB$ to the first member of this inequality, and its equal $\angle AOD$ to the second member, we have

$$\angle AOB + \angle BOC > \angle AOD + \angle COD.$$

$$\therefore \angle AOB + \angle BOC > \angle AOC.$$

PROP. XXXI. THEOREM.

458. *The sum of the face angles of any convex polyedral angle is less than four right angles.*



Given O–$ABCDE$ a convex polyedral $\angle$.

To Prove $\angle AOB + \angle BOC +$ etc. < 4 rt. $\angle$s.

Proof. Let $ABCDE$ be the base of the polyedral $\angle$.

Let O' be any point within polygon $ABCDE$, and draw lines $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.

Then, in triedral $\angle A$–EOB,

$$\angle OAE + \angle OAB > \angle O'AE + \angle O'AB. \qquad (§\ 457)$$

Also, $\angle OBA + \angle OBC > \angle O'BA + \angle O'BC$; etc.

Adding these inequalities, we have the sum of the base $\angle$s of the △s whose common vertex is $O >$ the sum of the base $\angle$s of the △s whose common vertex is O'.

But the sum of *all* the $\angle$s of the △s whose common vertex is O is equal to the sum of *all* the $\angle$s of the △s whose common vertex is O'. (§ 84)

Hence, the sum of the $\angle$s at O is $<$ the sum of the $\angle$s at O'.

Then, the sum of the $\angle$s at O is < 4 rt. $\angle$s. (§ 35)

PROP. XXXII. THEOREM.

459. *If two triedral angles have the face angles of one equal respectively to the face angles of the other, their homologous diedral angles are equal.*

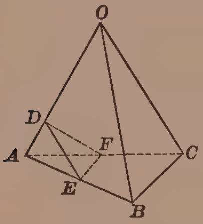


Fig. 1.

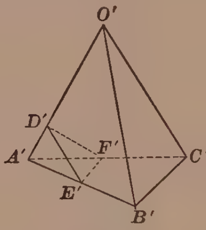


Fig. 2.

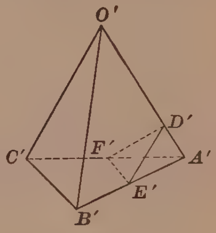


Fig. 3.

Given, in triedral $\angle$s $O\text{–}ABC$ and $O'\text{–}A'B'C'$,

$$\angle AOB = \angle A'O'B',\ \angle BOC = \angle B'O'C',$$
$$\text{and } \angle COA = \angle C'O'A'.$$

To Prove diedral $\angle OA =$ diedral $\angle O'A'$.

Proof. Lay off OA, OB, OC, $O'A'$, $O'B'$, and $O'C'$ all equal, and draw lines AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

$\therefore \triangle OAB = \triangle O'A'B'$. (§ 63)

$\therefore AB = A'B'$. (§ 66)

Similarly, $BC = B'C'$ and $CA = C'A'$.

$\therefore \triangle ABC = \triangle A'B'C'$. (§ 69)

$\therefore \angle EAF = \angle E'A'F'$. (?)

On OA and $O'A'$ take $AD = A'D'$.

Draw line DE in face $OAB \perp OA$.

Since $\triangle OAB$ is isosceles, $\angle OAB$ is acute, and hence DE will meet AB; let it meet AB at E.

Also, draw line DF in face $OAC \perp OA$, meeting AC at F; and lines $D'E'$ and $D'F'$ in faces $O'A'B'$ and $O'A'C' \perp O'A'$, meeting $A'B'$ and $A'C'$ at E' and F', respectively.

Draw lines EF and $E'F'$.

Then, in rt. $\triangle$s ADE and $A'D'E'$,

$$AD = A'D'.$$

And since $\triangle OAB = \triangle O'A'B'$,

$\angle DAE = \angle D'A'E'$. (?)

$\therefore \triangle ADE = \triangle A'D'E'$. (§ 89)

$\therefore AE = A'E'$, and $DE = D'E'$. (?)

Similarly, $AF = A'F'$, and $DF = D'F'$.

Then, in $\triangle$s AEF and $A'E'F'$,

$AE = A'E'$, $AF = A'F'$, and $\angle EAF = \angle E'A'F'$.

$\therefore \triangle AEF = \triangle A'E'F'$. (?)

$\therefore EF = E'F'$. (?)

Then, in $\triangle$s DEF and $D'E'F'$,

$DE = D'E'$, $DF = D'F'$, and $EF = E'F'$.

$\therefore \triangle DEF = \triangle D'E'F'$. (?)

$\therefore \angle EDF = \angle E'D'F'$. (?)

But, EDF and $E'D'F'$ are the plane $\angle$s of diedral $\angle$s OA and $O'A'$, respectively. (§ 429)

$\therefore$ diedral $\angle OA =$ diedral $\angle O'A'$. (§ 433)

Note. The above proof holds for Fig. 3 as well as for Fig. 2; in Figs. 1 and 2, the equal parts occur in the *same* order, and in Figs. 1 and 3 in the *reverse* order.

460. Cor. *If two triedral angles have the face angles of one equal respectively to the face angles of the other,*

1. *They are equal if the equal parts occur in the same order.*

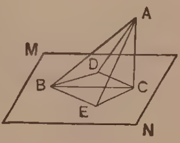
For if triedral $\angle O'\text{–}A'B'C'$ (Fig. 2) be applied to $O\text{–}ABC$ so that diedral $\angle$s $O'A'$ and OA coincide, point O' falling at O, then since $\angle A'O'C' = \angle AOC$, and $\angle A'O'B' = \angle AOB$, $O'B'$ will coincide with OB, and $O'C'$ with OC.

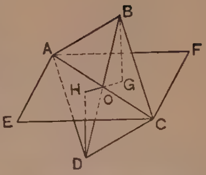
2. *They are symmetrical if the equal parts occur in the reverse order.*

EXERCISES.

18. If BC is the projection of line AB upon plane MN, and BD and BE be drawn in the plane making $\angle CBD = \angle CBE$, prove $\angle ABD = \angle ABE$.

(Lay off $BD = BE$, and draw lines AD, AE, CD, and CE. Prove $\triangle$s ABD and ABE equal.)

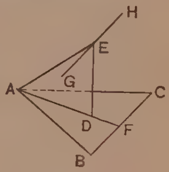




19. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

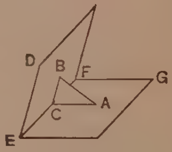
(Given plane *EF* through diagonal *AC* of ▱ *ABCD*; to prove *BG* = *DH*. Prove rt. △ *BGO* and *DHO* equal.)

20. Two triedral angles are equal when a face angle and the adjacent diedral angles of one are equal respectively to a face angle and the adjacent diedral angles of the other, and similarly placed.

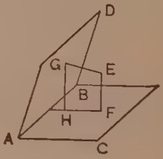


21. *D* is any point in perpendicular *AF* from *A* to side *BC* of triangle *ABC*. If line *DE* be drawn perpendicular to the plane of *ABC*, and line *GH* through *E* parallel to *BC*, prove line *AE* perpendicular to *GH*.

(Prove *BC* ⊥ to plane *AED* by § 438.)

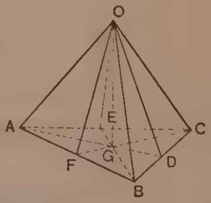


22. *A* is any point in face *EG* of diedral ∠ *DEFG*. If *AC* be drawn perpendicular to edge *EF*, and *AB* perpendicular to face *DF*, prove the plane determined by *AC* and *BC* perpendicular to *EF*. (Ex. 9.)



23. From any point *E* within diedral ∠ *CABD*, *EF* and *EG* are drawn perpendicular to faces *ABC* and *ABD*, respectively, and *GH* perpendicular to face *ABC* at *H*. Prove *FH* perpendicular to *AB*.

(Prove that *FH* lies in the plane of *EF* and *EG*.)



24. The three planes bisecting the diedral angles of a triedral angle meet in a common straight line.

(Let planes *OAD* and *OBE* intersect in line *OG*. Prove *G* in plane *OCF* by § 444.)

25. Any point in the plane passing through the bisector of an angle, perpendicular to its plane, is equally distant from the sides of the angle.

26. Any face angle of a polyedral angle is less than the sum of the remaining face angles.

(Divide the polyedral ∠ into triedral ∠s by passing planes through any lateral edge.)

Book VII.

POLYEDRONS.

DEFINITIONS.

461. A *polyedron* is a solid bounded by polygons.

The bounding polygons are called the *faces* of the polyedron; their sides are called the *edges*, and their vertices the *vertices*.

A *diagonal* of a polyedron is a straight line joining any two vertices not in the same face.

462. The least number of planes which can form a polyedral angle is three.

Whence, the least number of polygons which can bound a polyedron is four.

A polyedron of four faces is called a *tetraedron;* of six faces, a *hexaedron;* of eight faces, an *octaedron;* of twelve faces, a *dodecaedron;* of twenty faces, an *icosaedron.*

463. A polyedron is called *convex* when the section made by any plane is a convex polygon (§ 121).

All polyedrons considered hereafter will be understood to be convex.

464. The *volume* of a solid is its ratio to another solid, called the *unit of volume,* adopted arbitrarily as the unit of measure (§ 180).

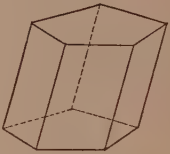
The usual unit of volume is a cube (§ 474) whose edge is some linear unit; for example, a *cubic inch* or a *cubic foot.*

465. Two solids are said to be *equivalent* when their volumes are equal.

PRISMS AND PARALLELOPIPEDS.

DEFINITIONS.

466. A *prism* is a polyedron, two of whose faces are equal polygons lying in parallel planes, having their homologous sides parallel, the other faces being parallelograms (§ 110).



The equal and parallel faces are called the *bases* of the prism, and the other faces the *lateral faces;* the edges which are not sides of the bases are called the *lateral edges,* and the sum of the areas of the lateral faces the *lateral area.*

The *altitude* is the perpendicular distance between the planes of the bases.

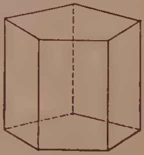
467. The following is given for convenience of reference:

The bases of a prism are equal.

468. It follows from the definition of § 466 that *the lateral edges of a prism are equal and parallel.* (§ 106, I)

469. A prism is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

470. A *right prism* is a prism whose lateral edges are perpendicular to its bases.

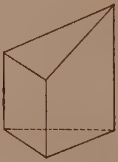


The lateral faces are rectangles (§ 398).

An *oblique prism* is a prism whose lateral edges are not perpendicular to its bases.

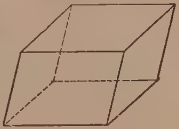
471. A *regular prism* is a right prism whose base is a regular polygon.

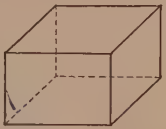
472. A *truncated prism* is a portion of a prism included between the base, and a plane, not parallel to the base, cutting all the lateral edges.



The base of the prism and the section made by the plane are called the *bases* of the truncated prism.

473. A *right section* of a prism is a section made by a plane cutting all the lateral edges, and perpendicular to them.





474. A *parallelopiped* is a prism whose bases are parallelograms; that is, all the faces are parallelograms.

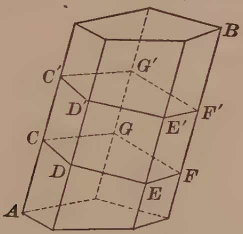
A *right parallelopiped* is a parallelopiped whose lateral edges are perpendicular to its bases.

A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles; that is, all the faces are rectangles.

A *cube* is a rectangular parallelopiped whose six faces are all squares.

PROP. I. THEOREM.

475. *The sections of a prism made by two parallel planes which cut all the lateral edges, are equal polygons.*



Given $\parallel$ planes CF and $C'F'$ cutting all the lateral edges of prism AB.

To Prove section $CDEFG =$ section $C'D'E'F'G'$.

Proof. We have $CD \parallel C'D'$, $DE \parallel D'E'$, etc. (§ 414)

$\therefore\ CD = C'D'$, $DE = D'E'$, etc. (§ 107)

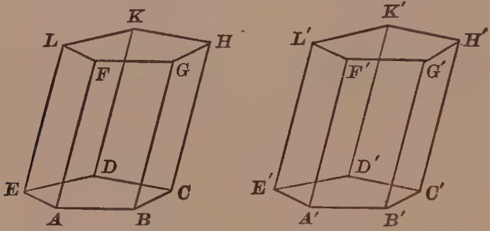
Also $\angle CDE = \angle C'D'E'$, $\angle DEF = \angle D'E'F'$, etc. (§ 426)

Then, polygons $CDEFG$ and $C'D'E'F'G'$, being mutually equilateral and mutually equiangular, are equal. (§ 124)

476. Cor. *The section of a prism made by a plane parallel to the base is equal to the base.*

PROP. II. THEOREM.

477. *Two prisms are equal when the faces including a triedral angle of one are equal respectively to the faces including a triedral angle of the other, and similarly placed.*



Given, in prisms AH and $A'H'$, faces $ABCDE$, AG, and AL equal respectively to faces $A'B'C'D'E'$, $A'G'$, and $A'L'$; the equal parts being similarly placed.

To Prove prism AH = prism $A'H'$.

Proof. We have $\measuredangle$s EAB, EAF, and FAB equal respectively to $\measuredangle$s $E'A'B'$, $E'A'F'$, and $F'A'B'$. (§ 66)

$\therefore$ triedral $\angle A\text{–}BEF$ = triedral $\angle A'\text{–}B'E'F'$. (§ 460, 1)

Then, prism $A'H'$ may be applied to prism AH in such a way that vertices A', B', C', D', E', G', F', and L' shall fall at A, B, C, D, E, G, F, and L, respectively.

Now since the lateral edges of the prisms are $\parallel$, edge $C'H'$ will fall on CH, $D'K'$ on DK, etc. (§ 53)

And since points G', F', and L' fall at G, F, and L, respectively, planes LH and $L'H'$ coincide. (§ 395, II)

Then points H' and K' fall at H and K, respectively.

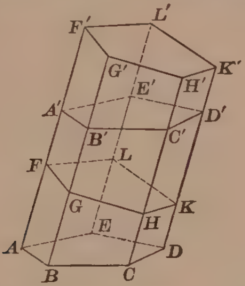
Hence, the prisms coincide throughout, and are equal.

478. Cor. *Two right prisms are equal when they have equal bases and equal altitudes*; for by inverting one of the prisms if necessary, the equal faces will be similarly placed.

479. Sch. The demonstration of § 477 applies without change to the case of two *truncated prisms.*

Prop. III. Theorem.

480. *An oblique prism is equivalent to a right prism, having for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*



Given FK' a right prism, having for its base FK a right section of oblique prism AD', and its altitude FF' equal to AA', a lateral edge of AD'.

To Prove $AD' \Leftrightarrow FK'$.

Proof. In truncated prisms AK and $A'K'$, faces $FGHKL$ and $F'G'H'K'L'$ are equal. (§ 475)

Therefore, $A'K'$ may be applied to AK so that vertices F', G', etc., shall fall at F, G, etc., respectively.

Then, edges $A'F'$, $B'G'$, etc., will coincide in direction with AF, BG, etc., respectively. (§ 399)

But since, by hyp., $FF' = AA'$, we have $AF = A'F'$.

In like manner, $BG = B'G'$, $CH = C'H'$, etc.

Hence, vertices A', B', etc., will fall at A, B, etc., respectively.

Then, $A'K'$ and AK coincide throughout, and are equal.

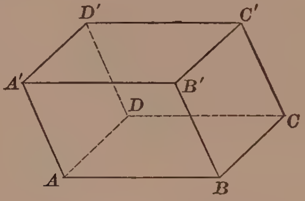
Now taking from the entire solid AK' truncated prism $A'K'$, there remains prism AD'.

And taking its equal AK, there remains prism FK'.

$$\therefore AD' \Leftrightarrow FK'.$$

PROP. IV. THEOREM.

481. *The opposite lateral faces of a parallelopiped are equal and parallel.*



Given AC and $A'C'$ the bases of parallelopiped AC'.

To Prove faces AB' and DC' equal and $\parallel$.

Proof. AB is equal and $\parallel$ to DC, and AA' to DD'. (§ 106, I)

$\therefore \angle A'AB = \angle D'DC$, and $AB' \parallel DC'$. (§ 426)

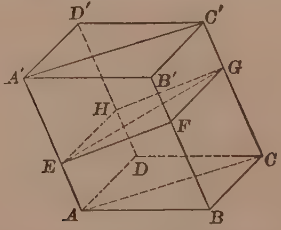
$\therefore$ face $AB' =$ face DC'. (§ 113)

Similarly, we may prove AD' and BC' equal and $\parallel$.

482. Cor. *Either face of a parallelopiped may be taken as the base.*

PROP. V. THEOREM.

483. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Given plane AC' passing through edges AA' and CC' of parallelopiped $A'C$.

To Prove prism $ABC\text{–}A' \Bumpeq$ prism $ACD\text{–}A'$.

Proof. Let $EFGH$ be a right section of the parallelopiped, intersecting plane $AA'C'C$ in line EG.

Now, face $AB' \parallel$ face DC'. (§ 481)

$\therefore EF \parallel GH$. (§ 414)

In like manner, $EH \parallel FG$, and $EFGH$ is a ▱.

$\therefore \triangle EFG = \triangle EGH$. (§ 108)

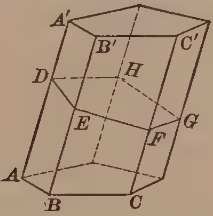
Now, $ABC\text{–}A'$ is $\Bumpeq$ a right prism whose base is EFG and altitude AA', and $ACD\text{–}A'$ is $\Bumpeq$ a right prism whose base is EGH and altitude AA'. (§ 480)

But these right prisms are equal, for they have equal bases and the same altitude. (§ 478)

$\therefore ABC\text{–}A' \Bumpeq ACD\text{–}A'$.

Prop. VI. Theorem.

484. *The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.*



Given $DEFGH$ a right section of prism AC'.

To Prove lat. area $AC' = (DE + EF + \text{etc.}) \times AA'$.

Proof. We have, $AA' \perp DE$. (§ 398)

$\therefore$ area $AA'B'B = DE \times AA'$. (§ 309)

Similarly, area $BB'C'C = EF \times BB'$

$= EF \times AA'$; etc. (§ 468)

Adding these equations, we have

$$\text{lat. area } AC' = DE \times AA' + EF \times AA' + \text{etc.}$$
$$= (DE + EF + \text{etc.}) \times AA'.$$

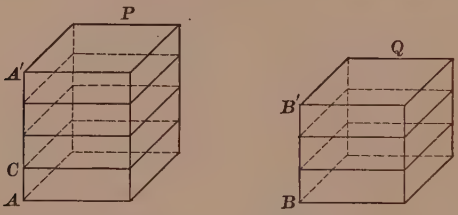
485. Cor. *The lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude.*

PROP. VII. THEOREM.

486. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Note. The phrase "rectangular parallelopiped" in the above statement signifies the *volume* of the rectangular parallelopiped.

CASE I. *When the altitudes are commensurable.*



Given P and Q rect. parallelopipeds, with equal bases, and commensurable altitudes, AA' and BB'.

To Prove $$\frac{P}{Q}=\frac{AA'}{BB'}.$$

Proof. Let AC be a common measure of AA' and BB', and suppose it to be contained 4 times in AA', and 3 times in BB'.

$$\therefore \frac{AA'}{BB'}=\frac{4}{3}. \quad (1)$$

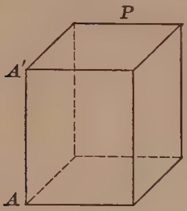
Through the several points of division of AA' and BB' pass planes $\perp$ to lines AA' and BB', respectively.

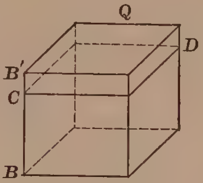
Then, rect. parallelopiped P will be divided into 4 parts, and rect. parallelopiped Q into 3 parts, all of which parts will be equal. (§ 478)

$$\therefore \frac{P}{Q}=\frac{4}{3}. \quad (2)$$

From (1) and (2), $$\frac{P}{Q}=\frac{AA'}{BB'}. \quad (?)$$

Case II. *When the altitudes are incommensurable.*





Given P and Q rect. parallelopipeds, with equal bases, and incommensurable altitudes, AA' and BB'.

To Prove $$\frac{P}{Q}=\frac{AA'}{BB'}.$$

Proof. Divide AA' into any number of equal parts, and apply one of these parts to BB' as a unit of measure.

Since AA' and BB' are incommensurable, a certain number of the parts will extend from B to C, leaving a remainder $CB' <$ one of the parts.

Draw plane $CD \perp BB'$, and let rect. parallelopiped BD be denoted by Q'.

Then since, by const., AA' and BC are commensurable,

$$\frac{P}{Q'}=\frac{AA'}{BC}. \qquad \text{(§ 486, Case I)}$$

Now let the number of subdivisions of AA' be indefinitely increased.

Then the length of each part will be indefinitely diminished, and remainder CB' will approach the limit 0.

Then, $\frac{P}{Q'}$ will approach the limit $\frac{P}{Q}$,

and $\frac{AA'}{BC}$ will approach the limit $\frac{AA'}{BB'}$.

$$\therefore \frac{P}{Q}=\frac{AA'}{BB'}. \qquad \text{(§ 188)}$$

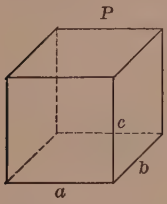
487. Def. The *dimensions* of a rectangular parallelopiped are the three edges which meet at any vertex.

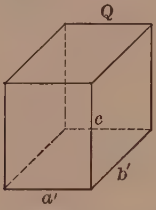
488. Sch. The theorem of § 486 may be expressed:

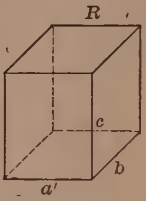
If two rectangular parallelopipeds have two dimensions of one equal respectively to two dimensions of the other, they are to each other as their third dimensions.

Prop. VIII. Theorem.

489. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*







Given P and Q rect. parallelopipeds, with the same altitude c, and the dimensions of the bases a, b, and a', b', respectively.

To Prove $$\frac{P}{Q}=\frac{a \times b}{a' \times b'}. \qquad (§ 305)$$

Proof. Let R be a rect. parallelopiped with the altitude c, and the dimensions of the base a' and b.

Then since P and R have each the dimensions b and c, they are to each other as their third dimensions a and a'. (§ 488)

That is, $$\frac{P}{R}=\frac{a}{a'}. \qquad (1)$$

And since R and Q have each the dimensions a' and c,

$$\frac{R}{Q}=\frac{b}{b'}. \qquad (2)$$

Multiplying (1) and (2), we have

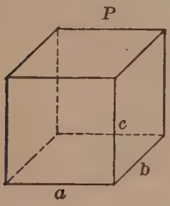
$$\frac{P}{R} \times \frac{R}{Q}, \text{ or } \frac{P}{Q}=\frac{a \times b}{a' \times b'}.$$

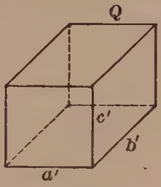
490. Sch. The theorem of § 489 may be expressed:

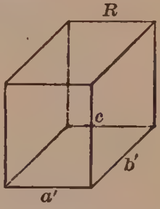
Two rectangular parallelopipeds having a dimension of one equal to a dimension of the other, are to each other as the products of their other two dimensions.

Prop. IX. Theorem.

491. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*







Given P and Q rect. parallelopipeds with the dimensions a, b, c, and a', b', c', respectively.

To Prove $$\frac{P}{Q}=\frac{a\times b\times c}{a'\times b'\times c'}.$$

(Let R be a rect. parallelopiped with the dimensions a', b', and c, and find values of $\frac{P}{R}$ and $\frac{R}{Q}$ by §§ 490 and 488.)

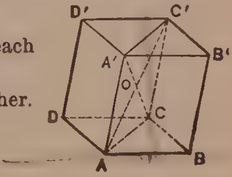
EXERCISES.

1. Two rectangular parallelopipeds, with equal altitudes, have the dimensions of their bases 6 and 14, and 7 and 9, respectively. Find the ratio of their volumes.

2. Find the ratio of the volumes of two rectangular parallelopipeds, whose dimensions are 8, 12, and 21, and 14, 15, and 24, respectively.

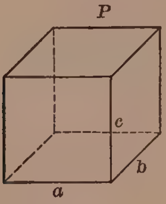
3. The diagonals of a parallelopiped bisect each other.

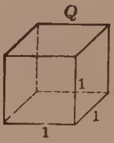
(To prove that AC' and $A'C$ bisect each other. Prove $AA'C'C$ a ▱ by § 110.)



Prop. X. Theorem.

492. *If the unit of volume is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped is equal to the product of its three dimensions.*





Given a, b, and c the dimensions of rect. parallelopiped P, and Q the unit of volume; that is, a cube whose edge is the linear unit.

To Prove vol. $P = a \times b \times c$.

Proof. We have

$$\frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1} \qquad (\S\ 491)$$

$$= a \times b \times c.$$

But since Q is the unit of volume,

$$\frac{P}{Q} = \text{vol. } P. \qquad (\S\ 464)$$

$$\therefore \text{vol. } P = a \times b \times c.$$

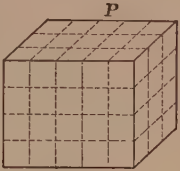
493. Sch. I. In all succeeding theorems relating to volumes, it is understood that the *unit of volume* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit. (Compare § 306.)

494. Cor. I. *The volume of a cube is equal to the cube of its edge.*

495. Cor. II. *The volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

(The proof is left to the pupil.)

496. Sch. II. If the dimensions of the rectangular parallelopiped are *multiples* of the linear unit, the truth of Prop. X. may be seen by dividing the solid into cubes, each equal to the unit of volume.



Thus, if the dimensions of rectangular parallelopiped P are 5 units, 4 units, and 3 units, respectively, the solid can evidently be divided into 60 cubes.

In this case, 60, the number which expresses the volume of the rectangular parallelopiped, is the product of 5, 4, and 3, the numbers which express the lengths of its edges.

EXERCISES.

4. Find the altitude of a rectangular parallelopiped, the dimensions of whose base are 21 and 30, equivalent to a rectangular parallelopiped whose dimensions are 27, 28, and 35.

5. Find the edge of a cube equivalent to a rectangular parallelopiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

6. Find the volume, and the area of the entire surface of a cube whose edge is $3\frac{1}{4}$ in.

7. Find the area of the entire surface of a rectangular parallelopiped, the dimensions of whose base are 11 and 13, and volume 858.

8. Find the volume of a rectangular parallelopiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

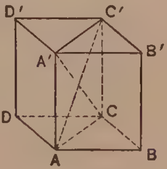
9. Find the dimensions of the base of a rectangular parallelopiped, the area of whose entire surface is 320, volume 336, and altitude 4.

(Represent the dimensions of the base by x and y.)

10. How many bricks, each 8 in. long, $2\frac{3}{4}$ in. wide, and 2 in. thick, will be required to build a wall 18 ft. long, 3 ft. high, and 11 in. thick?

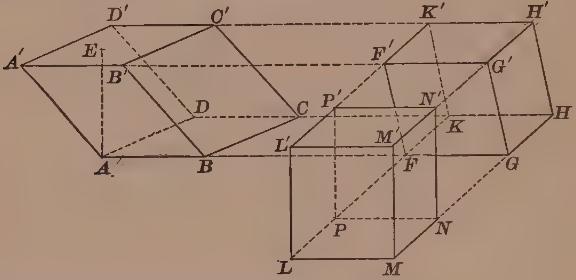
11. The diagonals of a rectangular parallelopiped are equal.

(Prove $AA'C'C$ a rectangle.)



PROP. XI. THEOREM.

497. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Given AE the altitude of parallelopiped AC'.

To Prove vol. $AC' =$ area $ABCD \times AE$.

Proof. Produce edges AB, $A'B'$, $D'C'$, and DC.

On AB produced, take $FG = AB$; and draw planes FK' and $GH' \perp FG$, forming right parallelopiped FH'.

$\therefore FH' \Leftrightarrow AC'$. (§ 480)

Produce edges HG, $H'G'$, $K'F'$, and KF.

On HG produced, take $NM = HG$; and draw planes NP' and $ML' \perp NM$, forming right parallelopiped LN'.

$\therefore LN' \Leftrightarrow FH'$. (§ 480)

$\therefore LN' \Leftrightarrow AC'$.

Now since, by cons., FG is $\perp$ plane GH', planes LH and MH' are $\perp$. (§ 441)

Then MM', being $\perp$ MN, is $\perp$ plane LH. (§ 438)

Whence, $\angle LMM'$ is a rt. $\angle$. (§ 398)

Then, LM' is a rectangle. (§ 76)

Therefore LN' is a rectangular parallelopiped.

$\therefore$ vol. $LN' =$ area $LMNP \times MM'$. (§ 495)

$\therefore$ vol. $AC' =$ area $LMNP \times MM'$. (1)

But rect. $LMNP =$ rect. $FGHK$; for they have equal bases MN and GH, and the same altitude. (§ 114)

Also, rect. $FGHK \Bumpeq$ ▱ $ABCD$; for they have equal bases FG and AB, and the same altitude. (§ 310)

$$\therefore LMNP \Bumpeq ABCD.$$

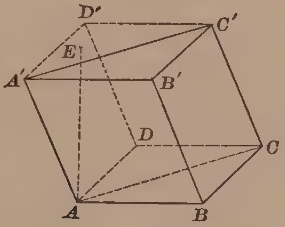
Also, $MM' = AE.$ (§ 424)

Substituting these values in (1), we have

$$\text{vol. } AC' = \text{area } ABCD \times AE.$$

Prop. XII. Theorem.

498. *The volume of a triangular prism is equal to the product of its base and altitude.*



Given AE the altitude of triangular prism $ABC\text{–}C'$.

To Prove vol. $ABC\text{–}C' =$ area $ABC \times AE$.

Proof. Construct parallelopiped $ABCD\text{–}D'$, having its edges ∥ to AB, BC, and BB', respectively.

$$\therefore \text{vol. } ABC\text{–}C' = \tfrac{1}{2} \text{ vol. } ABCD\text{–}D' \quad (§ 483)$$
$$= \tfrac{1}{2} \text{ area } ABCD \times AE \quad (§ 497)$$
$$= \text{area } ABC \times AE. \quad (§ 108)$$

EXERCISES.

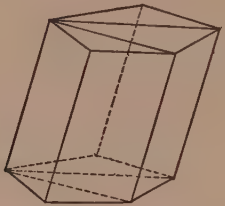
12. Find the lateral area and volume of a regular triangular prism, each side of whose base is 5, and whose altitude is 8.

13. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its dimensions.

(Fig. of Ex. 11. To prove $\overline{A'C}^2 = \overline{AA'}^2 + \overline{AB}^2 + \overline{AD}^2$.)

PROP. XIII. THEOREM.

499. *The volume of any prism is equal to the product of its base and altitude.*



Given any prism.

To Prove its volume equal to the product of its base and altitude.

Proof. The prism may be divided into triangular prisms by passing planes through one of the lateral edges and the corresponding diagonals of the base.

The volume of each triangular prism is equal to the product of its base and altitude. (§ 498)

Then, the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

Therefore, the volume of the given prism is equal to the product of its base and altitude.

500. Cor. I. *Two prisms having equivalent bases and equal altitudes are equivalent.*

501. Cor. II. 1. *Two prisms having equal altitudes are to each other as their bases.*

2. *Two prisms having equivalent bases are to each other as their altitudes.*

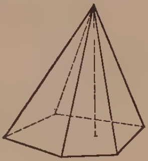
3. *Any two prisms are to each other as the products of their bases by their altitudes.*

Ex. 14. Find the lateral area and volume of a regular hexagonal prism, each side of whose base is 3, and whose altitude is 9.

PYRAMIDS.

DEFINITIONS.

502. A *pyramid* is a polyedron bounded by a polygon, called the *base*, and a series of triangles having a common vertex.



The common vertex of the triangular faces is called the *vertex* of the pyramid.

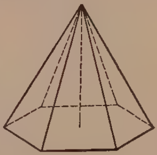
The triangular faces are called the *lateral faces*, and the edges terminating at the vertex the *lateral edges.*

The sum of the areas of the lateral faces is called the *lateral area.*

The *altitude* is the perpendicular distance from the vertex to the plane of the base.

503. A pyramid is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

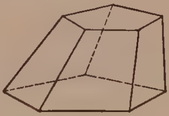
504. A *regular pyramid* is a pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular erected at the centre of the base.



505. A *truncated pyramid* is a portion of a pyramid included between the base and a plane cutting all the lateral edges.

The base of the pyramid and the section made by the plane are called the *bases* of the truncated pyramid.

506. A *frustum of a pyramid* is a truncated pyramid whose bases are parallel.



The *altitude* is the perpendicular distance between the planes of the bases.

EXERCISES.

15. Find the length of the diagonal of a rectangular parallelopiped whose dimensions are 8, 9, and 12.

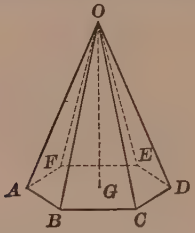
16. The diagonal of a cube is equal to its edge multiplied by $\sqrt{3}$.

PROP. XIV. THEOREM.

507. *In a regular pyramid,*

I. *The lateral edges are equal.*

II. *The lateral faces are equal isosceles triangles.*



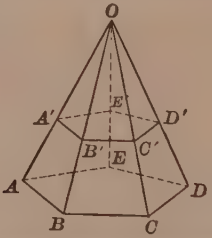
(The theorem follows by §§ 406, I, and 69.)

508. Def. The *slant height* of a regular pyramid is the altitude of any lateral face.

Or, it is the line drawn from the vertex of the pyramid to the middle point of any side of the base. (§ 94, I)

PROP. XV. THEOREM.

509. *The lateral faces of a frustum of a regular pyramid are equal trapezoids.*



Given AC' a frustum of regular pyramid $O\text{–}ABCDE$.

To Prove faces AB' and BC' equal trapezoids.

Proof. We have $\triangle OAB = \triangle OBC$. (§ 507, II)

We may then apply $\triangle OAB$ to $\triangle OBC$ in such a way that sides OB, OA, and AB shall coincide with sides OB, OC, and BC, respectively.

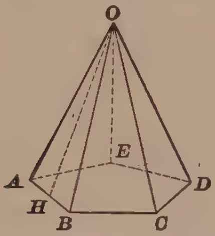
Now, $A'B' \parallel AB$ and $B'C' \parallel BC$. (?)
Hence, line $A'B'$ will coincide with line $B'C'$. (§ 53)
Then, AB' and BC' coincide throughout, and are equal.

510. Cor. *The lateral edges of a frustum of a regular pyramid are equal.*

511. Def. The *slant height* of a frustum of a regular pyramid is the altitude of any lateral face.

PROP. XVI. THEOREM.

512. *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.*



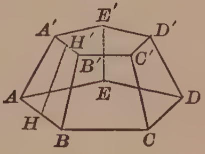
Given slant height OH of regular pyramid $O\text{–}ABCDE$.

To Prove

lat. area $O\text{–}ABCDE = (AB + BC + \text{etc.}) \times \frac{1}{2} OH$.

(By § 508, OH is the altitude of each lateral face.)

513. Cor. *The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases, multiplied by its slant height.*



Given slant height HH' of the frustum of a regular pyramid AD'.

To Prove

lat. area $AD' = \frac{1}{2}(AB + A'B' + BC + B'C' + \text{etc.}) \times HH'$.

(HH' is the altitude of each lateral face.)

EXERCISES.

17. The volume of a cube is $4\frac{17}{27}$ cu. ft. Find the area of its entire surface in square inches.

18. The volume of a right prism is 2310, and its base is a right triangle whose legs are 20 and 21, respectively. Find its lateral area.

19. Find the lateral area and volume of a right triangular prism, having the sides of its base 4, 7, and 9, respectively, and the altitude 8.

20. The volume of a regular triangular prism is $96\sqrt{3}$, and one side of its base is 8. Find its lateral area.

21. The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

(Represent the edge by x.)

22. A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain? (§§ 316, 499.)

23. The lateral area and volume of a regular hexagonal prism are 60 and $15\sqrt{3}$, respectively. Find its altitude, and one side of its base.

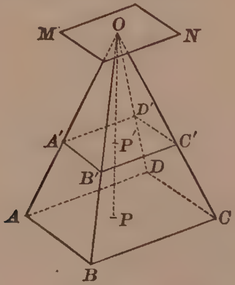
(Represent the altitude by x, and the side of the base by y.)

PROP. XVII. THEOREM.

514. *If a pyramid be cut by a plane parallel to its base,*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*



Given plane $A'C'$ ∥ to base of pyramid O–$ABCD$, cutting faces OAB, OBC, OCD, and ODA in lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, respectively, and altitude OP at P'.

I. **To Prove** $\frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC}$ etc. $=\frac{OP'}{OP}$.

Proof. Through O pass plane $MN \parallel ABCD$.

$$\therefore \frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC} \text{ etc.} =\frac{OP'}{OP}. \qquad (\S\ 427)$$

II. **To Prove** section $A'B'C'D'$ similar to $ABCD$.

Proof. We have $A'B' \parallel AB$, $B'C' \parallel BC$, etc. (?)

$\therefore \angle A'B'C' = \angle ABC$, $\angle B'C'D' = \angle BCD$, etc. (§ 426)

Again, $\triangle$ $OA'B'$, $OB'C'$, etc., are similar to $\triangle$ OAB, OBC, etc., respectively. (§ 257)

$$\therefore \frac{OA'}{OA}=\frac{A'B'}{AB},\ \frac{OB'}{OB}=\frac{B'C'}{BC} \text{ etc.} \qquad (1)$$

But, $$\frac{OA'}{OA}=\frac{OB'}{OB} \text{ etc.} \qquad (\S\ 514,\ \text{I})$$

$$\therefore \frac{A'B'}{AB}=\frac{B'C'}{BC}=\frac{C'D'}{CD} \text{ etc.} \qquad (?)$$

Then, polygons $A'B'C'D'$ and $ABCD$ are mutually equiangular, and have their homologous sides proportional.

Whence, $A'B'C'D'$ and $ABCD$ are similar. (§ 252)

515. Cor. I. Since $A'B'C'D'$ and $ABCD$ are similar,

$$\frac{\text{area } A'B'C'D'}{\text{area } ABCD}=\frac{\overline{A'B'}^2}{\overline{AB}^2}. \qquad (\S\ 322)$$

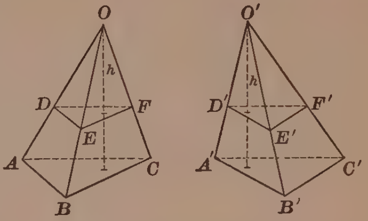
But from (1), § 514, $$\frac{A'B'}{AB}=\frac{OA'}{OA}$$

$$=\frac{OP'}{OP}. \qquad (\S\ 514,\ \text{I})$$

$$\therefore \frac{\text{area } A'B'C'D'}{\text{area } ABCD}=\frac{\overline{OP'}^2}{\overline{OP}^2}.$$

Hence, *the area of a section of a pyramid, parallel to the base, is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

516. Cor. II. *If two pyramids have equal altitudes and equivalent bases, sections parallel to the bases equally distant from the vertices are equivalent.*



Given bases of pyramids $O\text{–}ABC$ and $O'\text{–}A'B'C'$ ⇔, and the altitude of each pyramid $= H$; also DEF and $D'E'F'$ sections $=$ to the bases at distance h from O and O', respectively.

To Prove area $DEF =$ area $D'E'F'$.

Proof. We have

$$\frac{\text{area } DEF}{\text{area } ABC} = \frac{h^2}{H^2}, \text{ and } \frac{\text{area } D'E'F'}{\text{area } A'B'C'} = \frac{h^2}{H^2}. \quad (\S\ 515)$$

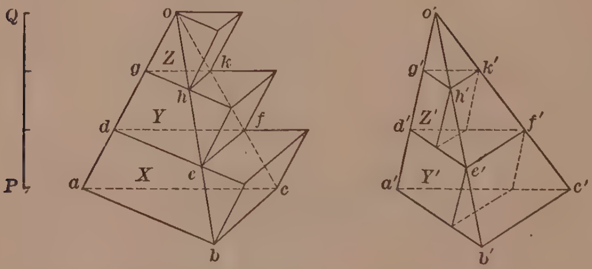
$$\therefore \frac{\text{area } DEF}{\text{area } ABC} = \frac{\text{area } D'E'F'}{\text{area } A'B'C'}. \quad (?)$$

But by hyp., area $ABC =$ area $A'B'C'$.

$$\therefore \text{area } DEF = \text{area } D'E'F'.$$

PROP. XVIII. THEOREM.

517. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



Given $o\text{–}abc$ and $o'\text{–}a'b'c'$ triangular pyramids with equal altitudes and ⇔ bases.

To Prove vol. $o\text{–}abc =$ vol. $o'\text{–}a'b'c'$.

Proof. Place the pyramids with their bases in the same plane, and let PQ be their common altitude.

Divide PQ into any number of equal parts.

Through the points of division pass planes ∥ to the plane of the bases, cutting $o\text{–}abc$ in sections def and ghk, and $o'\text{–}a'b'c'$ in sections $d'e'f'$ and $g'h'k'$, respectively.

$$\therefore\ def \bumpeq d'e'f', \text{ and } ghk \bumpeq g'h'k'. \qquad (\S\ 516)$$

With abc, def, and ghk as *lower* bases, construct prisms X, Y, and Z, with their lateral edges equal and ∥ to ad; and with $d'e'f'$ and $g'h'k'$ as *upper* bases, construct prisms Y' and Z', with their lateral edges equal and ∥ to $a'd'$.

$$\therefore \text{ prism } Y \bumpeq \text{ prism } Y', \text{ and prism } Z \bumpeq \text{ prism } Z'. \quad (\S\ 500)$$

Hence, the sum of the prisms circumscribed about $o\text{–}abc$ exceeds the sum of the prisms inscribed in $o'\text{–}a'b'c'$ by prism X.

But, $o\text{–}abc$ is evidently $<$ the sum of prisms X, Y, and Z; and it is $>$ the sum of prisms $\bumpeq$ to Y' and Z', respectively, which can be constructed with def and ghk as *upper* bases, having their lateral edges equal and ∥ to ad.

Again, $o'\text{–}a'b'c'$ is $>$ the sum of prisms Y' and Z'; and it is $<$ the sum of prisms $\bumpeq$ to X, Y, and Z, respectively, which can be constructed with $a'b'c'$, $d'e'f'$, and $g'h'k'$ as *lower* bases, having their lateral edges equal and ∥ to $a'd'$.

That is, each pyramid is $<$ the sum of prisms X, Y, and Z, and $>$ the sum of prisms Y' and Z'; whence, the difference of the volumes of the pyramids must be $<$ the difference of the volumes of the two systems of prisms, or $<$ volume X.

Now by sufficiently increasing the number of subdivisions of PQ, the volume of prism X may be made $<$ any assigned volume, however small.

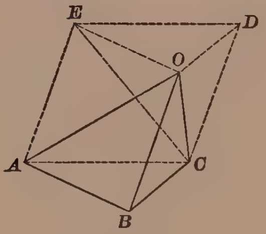
Hence, the volumes of the pyramids cannot differ by any volume, however small.

$$\therefore \text{ vol. } o\text{–}abc = \text{ vol. } o'\text{–}a'b'c'.$$

518. Cor. Since vol. $o'\text{–}a'b'c'$ is $>$ the total volume of the inscribed prisms, and $<$ the total volume of the circumscribed, the difference between vol. $o'\text{–}a'b'c'$ and the total volume of the inscribed prisms is $<$ the difference between the total volumes of the two systems of prisms, or $<$ vol. X; and hence approaches the limit 0 when the number of subdivisions is indefinitely increased.

PROP. XIX. THEOREM.

519. *A triangular pyramid is equivalent to one-third of a triangular prism having the same base and altitude.*



Given triangular pyramid $O\text{–}ABC$, and triangular prism $ABC\text{–}ODE$ having the same base and altitude.

To Prove vol. $O\text{–}ABC = \frac{1}{3}$ vol. $ABC\text{–}ODE$.

Proof. Prism $ABC\text{–}ODE$ is composed of triangular pyramid $O\text{–}ABC$, and quadrangular pyramid $O\text{–}ACDE$.

Divide the latter into two triangular pyramids, $O\text{–}ACE$ and $O\text{–}CDE$, by passing a plane through O, C, and E.

Now, $O\text{–}ACE$ and $O\text{–}CDE$ have the same altitude.

And since CE is a diagonal of $\square$ $ACDE$, they have equal bases, ACE and CDE. (§ 108)

$\therefore$ vol. $O\text{–}ACE =$ vol. $O\text{–}CDE$. (§ 517)

Again, pyramid $O\text{–}CDE$ may be regarded as having its vertex at C, and $\triangle$ ODE for its base.

Then, pyramids $O\text{–}ABC$ and $C\text{–}ODE$ have the same altitude. (§ 424)

They have also equal bases, ABC and ODE. (§ 467)

$$\therefore \text{vol. } O\text{–}ABC = \text{vol. } C\text{–}ODE. \quad (?)$$

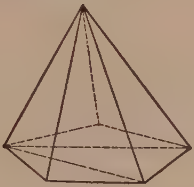
Then, vol. $O\text{–}ABC$ = vol. $O\text{–}ACE$ = vol. $O\text{–}CDE$. (?)

$$\therefore \text{vol. } O\text{–}ABC = \tfrac{1}{3} \text{ vol. } ABC\text{–}ODE.$$

520. Cor. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.* (§ 498)

Prop. XX. Theorem.

521. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



(Prove as in § 499.)

522. Cor. 1. *Two pyramids having equivalent bases and equal altitudes are equivalent.*

2. *Two pyramids having equal altitudes are to each other as their bases.*

3. *Two pyramids having equivalent bases are to each other as their altitudes.*

4. *Any two pyramids are to each other as the products of their bases by their altitudes.*

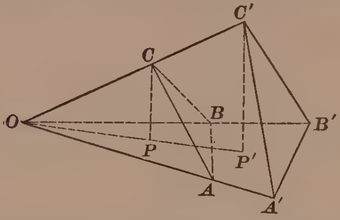
EXERCISES.

24. The altitude of a pyramid is 12 in., and its base is a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.? (§ 515.)

25. The altitude of a pyramid is 20 in., and its base is a rectangle whose dimensions are 10 in. and 15 in., respectively. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

PROP. XXI. THEOREM.

523. *Two tetraedrons having a triedral angle of one equal to a triedral angle of the other, are to each other as the products of the edges including the equal triedral angles.*



Given V and V' the volumes of tetraedrons $O\text{–}ABC$ and $O\text{–}A'B'C'$, respectively, having the common triedral $\angle O$.

To Prove $$\frac{V}{V'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Proof. Draw lines CP and $C'P' \perp$ to face $OA'B'$.

Let their plane intersect face $OA'B'$ in line OPP'.

Now, OAB and $OA'B'$ are the bases, and CP and $C'P'$ the altitudes, of triangular pyramids $C\text{–}OAB$ and $C'\text{–}OA'B'$, respectively.

$$\therefore \frac{V}{V'} = \frac{\text{area } OAB \times CP}{\text{area } OA'B' \times C'P'} \qquad (\S\ 522,\ 4)$$

$$= \frac{\text{area } OAB}{\text{area } OA'B'} \times \frac{CP}{C'P'}. \qquad (1)$$

But, $$\frac{\text{area } OAB}{\text{area } OA'B'} = \frac{OA \times OB}{OA' \times OB'}. \qquad (\S\ 321)$$

Also, $\triangle$ OCP and $OC'P'$ are rt. $\triangle$. (§ 398)

Then, $\triangle$ OCP and $OC'P'$ are similar. (§ 256)

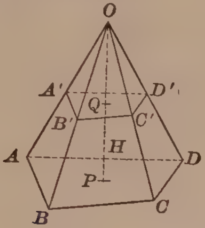
$$\therefore \frac{CP}{C'P'} = \frac{OC}{OC'}. \qquad (?)$$

Substituting these values in (1), we have

$$\frac{V}{V'} = \frac{OA \times OB}{OA' \times OB'} \times \frac{OC}{OC'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

PROP. XXII. THEOREM.

524. *The volume of a frustum of a pyramid is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Given B the area of the lower base, b the area of the upper base, and H the altitude, of AC', a frustum of any pyramid O–AC.

To Prove vol. $AC' = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H$. (§ 233)

Proof. Draw altitude OP, cutting $A'C'$ at Q.

Now, vol. $AC' =$ vol. O–$AC -$ vol. O–$A'C'$

$$= B \times \tfrac{1}{3}(H + OQ) - b \times \tfrac{1}{3} OQ \qquad (\S\ 521)$$

$$= B \times \tfrac{1}{3} H + B \times \tfrac{1}{3} OQ - b \times \tfrac{1}{3} OQ$$

$$= B \times \tfrac{1}{3} H + (B - b) \times \tfrac{1}{3} OQ. \qquad (1)$$

But, $$B : b = \overline{OP}^2 : \overline{OQ}^2. \qquad (\S\ 515)$$

Taking the square root of each term,

$$\sqrt{B} : \sqrt{b} = OP : OQ. \qquad (\S\ 241)$$

$$\therefore\ \sqrt{B} - \sqrt{b} : \sqrt{b} = OP - OQ : OQ \qquad (\S\ 238)$$

$$= H : OQ.$$

$$\therefore\ (\sqrt{B} - \sqrt{b}) \times OQ = \sqrt{b} \times H. \qquad (\S\ 232)$$

Multiplying both members by $(\sqrt{B} + \sqrt{b})$,

$$(B - b) \times OQ = (\sqrt{B \times b} + b) \times H.$$

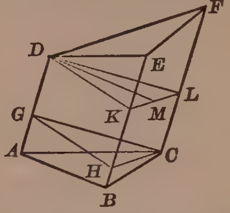
Substituting this value in (1), we have

$$\text{vol. } AC' = B \times \tfrac{1}{3} H + (\sqrt{B \times b} + b) \times \tfrac{1}{3} H$$

$$= (B + b + \sqrt{B \times b}) \times \tfrac{1}{3} H.$$

PROP. XXIII. THEOREM.

525. *The volume of a truncated triangular prism is equal to the product of a right section by one-third the sum of the lateral edges.*



Given GHC and DKL rt. sections of truncated triangular prism $ABC\text{–}DEF$.

To Prove

vol. $ABC\text{–}DEF =$ area $GHC \times \frac{1}{3}(AD + BE + CF)$.

Proof. Draw line $DM \perp KL$.

The given truncated prism consists of the rt. triangular prism $GHC\text{–}DKL$, and pyramids $D\text{–}EKLF$ and $C\text{–}ABHG$.

vol. $GHC\text{–}DKL =$ area $GHC \times GD$ (§ 498)

$=$ area $GHC \times \frac{1}{3}(GD + HK + CL)$, (1)

since the lateral edges of a prism are equal (§ 468).

Now DM is the altitude of pyramid $D\text{–}EKLF$. (§ 438)

$\therefore$ vol. $D\text{–}EKLF =$ area $EKLF \times \frac{1}{3} DM$. (§ 521)

But KL is the altitude of trapezoid $EKLF$. (§ 398)

$\therefore$ vol. $D\text{–}EKLF = \frac{1}{2}(KE + LF) \times KL \times \frac{1}{3} DM$. (§ 316)

Rearranging the factors, we have

vol. $D\text{–}EKLF = (\frac{1}{2} KL \times DM) \times \frac{1}{3}(KE + LF)$

$=$ area $DKL \times \frac{1}{3}(KE + LF)$ (§ 312)

$=$ area $GHC \times \frac{1}{3}(KE + LF)$. (2)

In like manner, we may prove

vol. $C\text{–}ABHG =$ area $GHC \times \frac{1}{3}(AG + BH)$. (3)

Adding (1), (2), and (3), the sum of the volumes of the solids $GHC\text{–}DKL$, $D\text{–}EKLF$, and $C\text{–}ABHG$ is

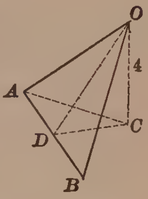
area $GHC \times \frac{1}{3}(\overline{AG + GD} + \overline{BH + HK + KE} + \overline{CL + LF})$.

$\therefore$ vol. $ABC\text{–}DEF =$ area $GHC \times \frac{1}{3}(AD + BE + CF)$.

526. Cor. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of the lateral edges.*

EXERCISES.

26. Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume.



Let OAB be a lateral face of the regular triangular pyramid, and C the centre of the base; draw line $CD \perp AB$; also, lines OC, AC, and OD.

Now, $AC = \frac{AB}{\sqrt{3}}$ (§ 356) $= \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

∴ lat. edge $OA = \sqrt{\overline{AC}^2 + \overline{OC}^2}$ (§ 272) $= \sqrt{12 + 16} = \sqrt{28} = 2\sqrt{7}$.

∴ slant ht. $OD = \sqrt{\overline{OA}^2 - \overline{AD}^2}$ (§ 273) $= \sqrt{28 - 9} = \sqrt{19}$.

∴ lat. area of pyramid $= 9\sqrt{19}$ (§ 512).

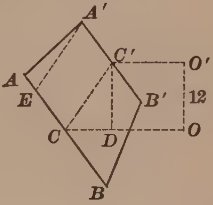
Again, $CD = \sqrt{\overline{AC}^2 - \overline{AD}^2} = \sqrt{12 - 9} = \sqrt{3}$.

∴ area of base $= \frac{1}{2} \times 18 \times \sqrt{3}$ (§ 350) $= 9\sqrt{3}$.

∴ vol. of pyramid $= \frac{1}{3} \times 9\sqrt{3} \times 4$ (§ 520) $= 12\sqrt{3}$.

27. Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17 and 7, respectively, and whose altitude is 12.

Let $ABB'A'$ be a lateral face of the frustum, and O and O' the centres of the bases; draw lines $OC \perp AB$, $O'C' \perp A'B'$, $C'D \perp OC$, and $A'E \perp AB$; also, lines OO' and CC'.



Now, $CD = OC - O'C' = 8\frac{1}{2} - 3\frac{1}{2} = 5$.

∴ Slant ht. CC'

$= \sqrt{\overline{CD}^2 + \overline{C'D}^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.

∴ lat. area frustum

$= \frac{1}{2}(68 + 28) \times 13$ (§ 513) $= 624$.

Again, $AE = AC - A'C' = 8\frac{1}{2} - 3\frac{1}{2} = 5$, and $A'E = CC' = 13$.

∴ lat. edge $AA' = \sqrt{\overline{AE}^2 + \overline{A'E}^2} = \sqrt{25 + 169} = \sqrt{194}$.

Again, area lower base $= 17^2$, area upper base $= 7^2$, and a mean proportional between them $= \sqrt{17^2 \times 7^2} = 17 \times 7 = 119$.

∴ vol. frustum $= (289 + 49 + 119) \times 4$ (§ 524) $= 1828$.

Find the lateral edge, lateral area, and volume

28. Of a regular triangular pyramid, each side of whose base is 12, and whose altitude is 15.

29. Of a regular quadrangular pyramid, each side of whose base is 3, and whose altitude is 5.

30. Of a regular hexagonal pyramid, each side of whose base is 4, and whose altitude is 9.

31. Of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, respectively, and whose altitude is 24.

32. Of a frustum of a regular quadrangular pyramid, the sides of whose bases are 9 and 5, respectively, and whose altitude is 10.

33. Of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, respectively, and whose altitude is 12.

34. Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

35. Find the volume of a truncated regular quadrangular prism, a side of whose base is 8, and whose lateral edges, taken in order, are 2, 6, 8, and 4, respectively.

(Pass a plane through two diagonally opposite lateral edges, dividing the solid into two truncated right triangular prisms.)

36. Find the volume of a truncated right triangular prism, whose lateral edges are 11, 14, and 17, having for its base an isosceles triangle whose sides are 10, 13, and 13, respectively.

37. The slant height and lateral edge of a regular quadrangular pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area and volume.

38. The altitude and slant height of a regular hexagonal pyramid are 15 and 17, respectively. Find its lateral edge and volume.

(Represent the side of the base by x.)

39. The lateral edge of a frustum of a regular hexagonal pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.

40. Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, respectively, and whose lateral edge is 5.

41. Find the lateral area and volume of a regular quadrangular pyramid, the area of whose base is 100, and whose lateral edge is 13.

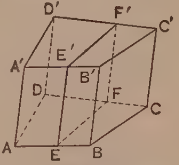
42. Prove the lateral surface of a pyramid greater than its base, when the perpendicular from the vertex to the base falls within the base.

(From foot of altitude draw lines to the vertices of the base; each △ formed has a smaller altitude than the corresponding lateral face.)

43. If E, F, G, and H are the middle points of edges AB, AD, CD, and BC, respectively, of tetraedron $ABCD$, prove $EFGH$ a parallelogram. (§ 130.)

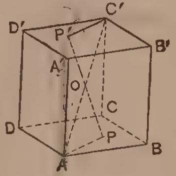
44. Two tetraedrons are equal if a diedral angle and the adjacent faces of one are equal, respectively, to a diedral angle and the adjacent faces of the other, if the equal parts are similarly placed.

(Figs. of § 459. Given faces OAB, OAC, and diedral $\angle OA$ equal, respectively, to faces $O'A'B'$, $O'A'C'$, and diedral $\angle O'A'$.)



45. The section of a prism made by a plane parallel to a lateral edge is a parallelogram.

(Given section $EE'F'F \parallel AA$. Prove $EE' \parallel$ to plane CD'; then use § 412.)



46. The point of intersection of the diagonals of a parallelopiped is called the *centre* of the parallelopiped. (Ex. 3.)

Prove that any line drawn through the centre of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

47. The volume of a regular prism is equal to its lateral area, multiplied by one-half the apothem of its base. (§ 350.)

48. The volume of a regular pyramid is equal to its lateral area, multiplied by one-third the distance from the centre of its base to any lateral face.

(Pass planes through the lateral edges and the centre of the base.)

49. Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.

50. The areas of the bases of a frustum of a pyramid are 12 and 75, respectively, and its altitude is 9. What is the altitude of the pyramid?

(Let altitude of pyramid $= x$; then $x - 9$ is the $\perp$ from its vertex to the upper base of the frustum; then use § 515.)

51. The bases of a frustum of a pyramid are rectangles, whose sides are 27 and 15, and 9 and 5, respectively, and the line joining their centres is perpendicular to each base. If the altitude of the frustum is 12, find its lateral area and volume.

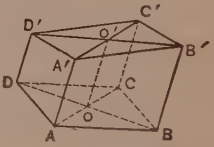
(From the centre of each base draw $\perp$s to two of its sides; in this way the altitudes of the lateral faces may be found.)

52. A frustum of any pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for their bases the lower base, the upper base, and a mean proportional between the bases, of the frustum. (§ 524.)

53. The upper base of a truncated parallelopiped is a parallelogram.

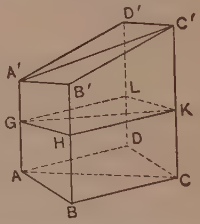
54. The sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

(Let planes AC' and BD' intersect in OO'. Find the length of OO' in terms of the lateral edges by § 132.)



55. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by one-fourth the sum of the lateral edges.

(By proof of § 483, a rt. section of a parallelopiped is a ▱; divide the solid into two truncated triangular prisms, and apply Ex. 54.)



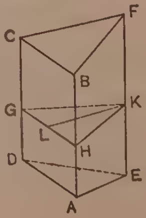
56. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by the distance between the centres of the bases.

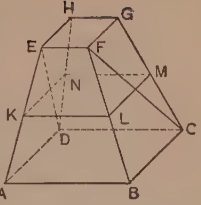
(By Ex. 54, the distance between the centres of the bases may be proved equal to one-fourth the sum of the lateral edges.)

57. If $ABCD$ is a rectangle, and EF any line not in its plane parallel to AB, the volume of the solid bounded by figures $ABCD$, $ABFE$, $CDEF$, ADE, and BCF, is

$$\tfrac{1}{6}\, h \times AD \times (2\,AB + EF),$$

where h is the perpendicular from any point of EF to $ABCD$. (§ 525.)





58. If $ABCD$ and $EFGH$ are rectangles lying in parallel planes, AB and BC being parallel to EF and FG, respectively, the solid bounded by the figures $ABCD$, $EFGH$, $ABFE$, $BCGF$, $CDHG$, and $DAEH$, is called a *rectangular prismoid.*

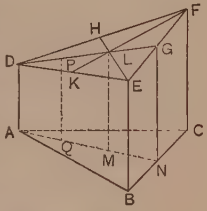
$ABCD$ and $EFGH$ are called the *bases* of the rectangular prismoid, and the perpendicular distance between them the *altitude.*

Prove the volume of a rectangular prismoid equal to the sum of its bases, plus four times a section equally distant from the bases, multiplied by one-sixth the altitude.

(Pass a plane through CD and EF, and find volumes of solids $ABCD$–EF and $EFGH$–CD by Ex. 57.)

59. Find the volume of rectangular prismoid the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

60. Two tetraedrons are equal if three faces of one are equal, respectively, to three faces of the other, if the equal parts are similarly placed. (§ 460, 1.)



61. The perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one-third the sum of the lateral edges.

(Let P be the middle point of DL, and draw $PQ \perp ABC$; express LM in terms of PQ and GN by § 132.)

62. The three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, meet in a common straight line.

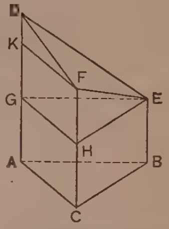
(Fig. of Ex. 24, p. 272. The intersections of the planes with the base of the pyramid are the medians of the base.)

63. A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular quadrangular pyramid 2 ft. in height, each side of whose base is 2 ft. What is its weight, at 180 lb. to the cubic foot?

64. Find the area of the base of a regular quadrangular pyramid, whose lateral faces are equilateral triangles, and whose altitude is 5.

(Represent lateral edge and side of base by x.)

65. A plane passed through the centre of a parallelopiped divides it into two equivalent solids. (Ex. 55.)

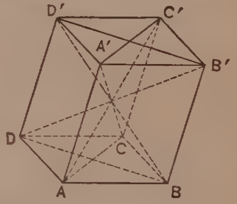


66. The sides of the base, *AB*, *BC*, and *CA*, of truncated right triangular prism *ABC–DEF* are 15, 4, and 12, respectively, and the lateral edges *AD*, *BE*, and *CF* are 15, 7, and 10, respectively. Find the area of upper base *DEF*.

(Draw $EH \perp CF$, and *HG* and $FK \perp AD$. Find area *DEF* by § 324.)

67. The volume of a triangular prism is equal to a lateral face, multiplied by one-half its perpendicular distance from any point in the opposite lateral edge.

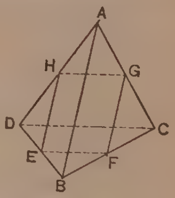
(Draw a rt. section of the prism, and apply § 525.)



68. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

(To prove $\overline{AC'}^2 + \overline{A'C}^2 + \overline{BD'}^2 + \overline{B'D}^2$ equal to $4\,\overline{AA'}^2 + 4\,\overline{AB}^2 + 4\,\overline{AD}^2$. Apply Ex. 79, p. 228, to ▱ *AA′C′C*.)

69. The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.



70. If *ABCD* is a tetraedron, the section made by a plane parallel to each of the edges *AB* and *CD* is a parallelogram. (§ 412.)

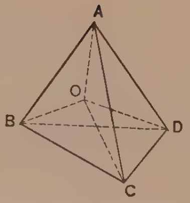
(To prove *EFGH* a ▱.)

71. In tetraedron *ABCD*, a plane is drawn through edge *CD* perpendicular to *AB*, intersecting faces *ABC* and *ABD* in *CE* and *ED*, respectively. If the bisector of ∠ *CED* meets *CD* at *F*, prove

$$CF : DF = \text{area } ABC : \text{area } ABD. \quad (\S\ 249.)$$

72. The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron (§ 536) is equal to its altitude.

(Divide the tetraedron into triangular pyramids, having the given point for their common vertex.)



73. The planes bisecting the diedral angles of a tetraedron intersect in a common point.

74. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

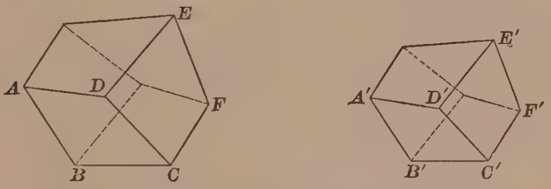
(In Fig. of Ex. 68, let AC', $A'C$, BD', and $B'D$ pass through a common point. To prove AC' a parallelopiped. Prove AC a ▱.)

SIMILAR POLYEDRONS.

527. Def. Two polyedrons are said to be *similar* when they have the same number of faces similar each to each and similarly placed, and have their homologous polyedral angles equal.

PROP. XXIV. THEOREM.

528. *The ratio of any two homologous edges of two similar polyedrons is equal to the ratio of any other two homologous edges.*



Given, in similar polyedrons AF and $A'F'$, edge AB homologous to edge $A'B'$, and edge EF to edge $E'F'$; and faces AC and DF similar to faces $A'C'$ and $D'F'$, respectively.

To Prove $\dfrac{AB}{A'B'} = \dfrac{EF}{E'F'}$.

$\left(\text{By § 253, 2, } \dfrac{AB}{A'B'} = \dfrac{CD}{C'D'}.\right)$

529. Cor. I. *Any two homologous faces of two similar polyedrons are to each other as the squares of any two homologous edges.*

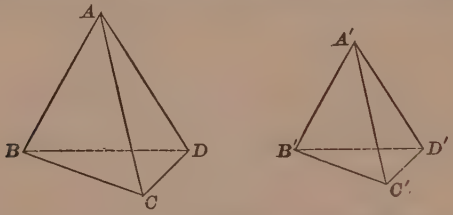
$$\left(\text{To prove } \frac{\text{area } ABCD}{\text{area } A'B'C'D'} = \frac{\overline{EF}^2}{\overline{E'F'}^2}. \quad \text{See § 322.}\right)$$

530. Cor. II. *The entire surfaces of two similar polyedrons are to each other as the squares of any two homologous edges.*

$$\left(\text{To prove } \frac{\text{area } ABCD + \text{area } CDEF \text{ etc.}}{\text{area } A'B'C'D' + \text{area } C'D'E'F' \text{ etc.}} = \frac{\overline{EF}^2}{\overline{E'F'}^2}.\right)$$

PROP. XXV. THEOREM.

531. *Two tetraedrons are similar when the faces including a triedral angle of one are similar, respectively, to the faces including a triedral angle of the other, and similarly placed.*



Given, in tetraedrons $ABCD$ and $A'B'C'D'$, face ABC similar to $A'B'C'$, ACD to $A'C'D'$, and ADB to $A'D'B'$.

To Prove $ABCD$ and $A'B'C'D'$ similar.

Proof. From the given similar faces, we have

$$\frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{BD}{B'D'}. \qquad (?)$$

Hence, faces BCD and $B'C'D'$ are similar. (§ 259)

Again, ∠s BAC, CAD, and DAB are equal, respectively, to ∠s $B'A'C'$, $C'A'D'$, and $D'A'B'$. (?)

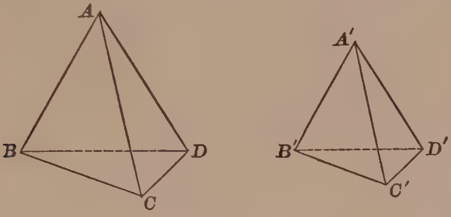
Then, triedral ∠s $A\text{–}BCD$ and $A'\text{–}B'C'D'$ are equal. (§ 460, 1)

Similarly, any two homologous triedral ∠s are equal.

Therefore, $ABCD$ and $A'B'C'D'$ are similar (§ 527).

Prop. XXVI. Theorem.

532. *Two tetraedrons are similar when a diedral angle of one is equal to a diedral angle of the other, and the faces including the equal diedral angles similar each to each, and similarly placed.*



Given, in tetraedrons $ABCD$ and $A'B'C'D'$, diedral $\angle AB$ equal to diedral $\angle A'B'$; and faces ABC and ABD similar to faces $A'B'C'$ and $A'B'D'$, respectively.

To Prove $ABCD$ and $A'B'C'D'$ similar.

Proof. Apply tetraedron $A'B'C'D'$ to $ABCD$ so that diedral $\angle A'B'$ shall coincide with its equal diedral $\angle AB$, point A' falling at A.

Then since $\angle B'A'C' = \angle BAC$ and $\angle B'A'D' = \angle BAD$, edge $A'C'$ will coincide with edge AC, and $A'D'$ with AD.

$$\therefore \angle C'A'D' = \angle CAD.$$

Again, from the given similar faces,

$$\frac{A'C'}{AC} = \frac{A'B'}{AB} = \frac{A'D'}{AD}. \qquad (?)$$

Hence, $\triangle C'A'D'$ is similar to $\triangle CAD$. (§ 261)

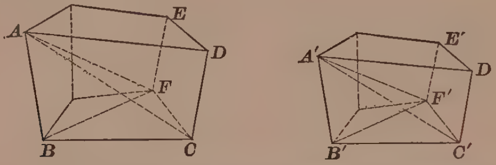
Then, the faces including triedral $\angle A'\text{–}B'C'D'$ are similar respectively to the faces including triedral $\angle A\text{–}BCD$, and similarly placed.

Therefore, $ABCD$ and $A'B'C'D'$ are similar. (§ 531)

Ex. 75. If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.

PROP. XXVII. THEOREM.

533. *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.*



Given AF and $A'F'$ similar polyedrons, vertices A and A' being homologous.

To Prove that they may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Proof. Divide all the faces of AF, except the ones having A as a vertex, into $\triangle$; and draw lines from A to their vertices.

In like manner, divide all the faces of $A'F'$, except the ones having A' as a vertex, into $\triangle$ similar to those in AF, and similarly placed. (§ 267)

Draw lines from A' to their vertices.

Then, the given polyedrons are decomposed into the same number of tetraedrons, similarly placed.

Let $ABCF$ and $A'B'C'F'$ be homologous tetraedrons.

$\triangle$ ABC and BCF are similar, respectively, to $\triangle$ $A'B'C'$ and $B'C'F'$. (§ 267)

And since the given polyedrons are similar, the homologous diedral $\angle$ BC and $B'C'$ are equal.

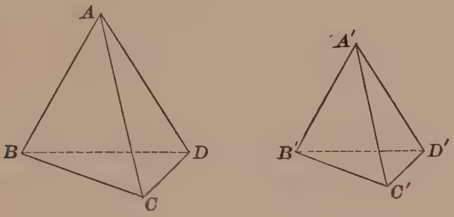
Therefore, $ABCF$ and $A'B'C'F'$ are similar. (§ 532)

In like manner, we may prove any two homologous tetraedrons similar.

Hence, the given polyedrons are decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Prop. XXVIII. Theorem.

534. *Two similar tetraedrons are to each other as the cubes of their homologous edges.*



Given V and V' the volumes of similar tetraedrons $ABCD$ and $A'B'C'D'$, vertices A and A' being homologous.

To Prove $$\frac{V}{V'}=\frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Proof. Since the triedral ∠s at A and A' are equal,

$$\frac{V}{V'}=\frac{AB\times AC\times AD}{A'B'\times A'C'\times A'D'} \qquad (\S\ 523)$$

$$=\frac{AB}{A'B'}\times\frac{AC}{A'C'}\times\frac{AD}{A'D'}.$$

But, $$\frac{AC}{A'C'}=\frac{AB}{A'B'},\ \text{and}\ \frac{AD}{A'D'}=\frac{AB}{A'B'}. \qquad (\S\ 528)$$

$$\therefore\ \frac{V}{V'}=\frac{AB}{A'B'}\times\frac{AB}{A'B'}\times\frac{AB}{A'B'}=\frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

535. Cor. *Any two similar polyedrons are to each other as the cubes of their homologous edges.*

For any two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each (§ 533).

Any two homologous tetraedrons are to each other as the cubes of their homologous edges. (§ 534)

Then, any two homologous tetraedrons are to each other as the cubes of any two homologous edges of the polyedrons. (§ 528)

REGULAR POLYEDRONS.

536. Def. A *regular polyedron* is a polyedron whose faces are equal regular polygons, and whose polyedral angles are all equal.

PROP. XXIX. THEOREM.

537. *Not more than five regular convex polyedrons are possible.*

A convex polyedral ∠ must have at least three faces, and the sum of its face ∠s must be < 360° (§ 458).

1. *With equilateral triangles.*

Since the ∠ of an equilateral △ is 60°, we may form a convex polyedral ∠ by combining either 3, 4, or 5 equilateral △s.

Not more than 5 equilateral △s can be combined to form a convex polyedral ∠. (§ 458)

Hence, not more than three regular convex polyedrons can be bounded by equilateral △s.

2. *With squares.*

Since the ∠ of a square is 90°, we may form a convex polyedral ∠ by combining 3 squares.

Not more than 3 squares can be combined to form a convex polyedral ∠. (?)

Hence, not more than one regular convex polyedron can be bounded by squares.

3. *With regular pentagons.*

Since the ∠ of a regular pentagon is 108°, we may form a convex polyedral ∠ by combining 3 regular pentagons.

Not more than 3 regular pentagons can be combined to form a convex polyedral ∠. (?)

Hence, not more than one regular convex polyedron can be bounded by regular pentagons.

Since the ∠ of a regular hexagon is 120°, no convex polyedral ∠ can be formed by combining regular hexagons. (?)

Hence, no regular convex polyedron can be bounded by regular hexagons.

In like manner, no regular convex polyedron can be bounded by regular polygons of more than six sides.

Therefore, not more than five regular convex polyedrons are possible.

Prop. XXX. Theorem.

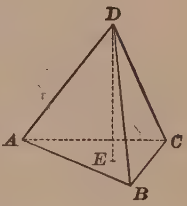
538. *With a given edge, to construct a regular polyedron.*

We will now prove, by actual construction, that five regular convex polyedrons are possible:

1. The regular tetraedron, bounded by 4 equilateral $\triangle$s.
2. The regular hexaedron, or cube, bounded by 6 squares.
3. The regular octaedron, bounded by 8 equilateral $\triangle$s.
4. The regular dodecaedron, bounded by 12 regular pentagons.
5. The regular icosaedron, bounded by 20 equilateral $\triangle$s.

1. *To construct a regular tetraedron.*

Given line AB.



Required to construct with AB as an edge a regular tetraedron.

Construction. Construct the equilateral $\triangle ABC$.

At its centre E, draw line $ED \perp ABC$; and take point D so that $AD = AB$.

Draw lines AD, BD, and CD.

Then, solid $ABCD$ is a regular tetraedron.

Proof. Since A, B, and C are equally distant from E,

$$AD = BD = CD. \qquad (\S\ 406,\ \text{I})$$

Hence, the six edges of the tetraedron are all equal.

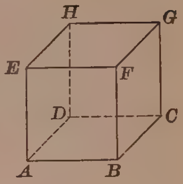
Then, the faces are equal equilateral $\triangle$s. (§ 69)

And since the $\angle$s of the faces are all equal, the triedral $\angle$s whose vertices are A, B, C, and D are all equal. (§ 460, 1)

Therefore, solid $ABCD$ is a regular tetraedron. (§ 536)

2. *To construct a regular hexaedron, or cube.*

Given line AB.



Required to construct with AB as an edge a cube.

Construction. Construct square $ABCD$; and draw lines AE, BF, CG, and DH, each equal to AB, and $\perp$ $ABCD$.

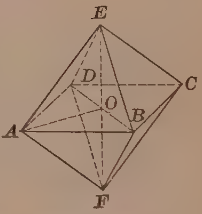
Draw lines EF, FG, GH, and HE; then, solid AG is a cube.

Proof. By cons., its faces are equal squares.

Hence, its triedral ∠s are all equal. (§ 460, 1)

3. *To construct a regular octaedron.*

Given line AB.



Required to construct with AB as an edge a regular octaedron.

Construction. Construct the square $ABCD$; through its centre O draw line $EF \perp ABCD$, making $OE = OF = OA$.

Draw lines EA, EB, EC, ED, FA, FB, FC, and FD; then solid $AEFC$ is a regular octaedron.

Proof. Draw lines OA, OB, and OD.

Then in rt. △s AOB, AOE, and AOF, by cons.,

$$OA = OB = OE = OF.$$

$$\therefore \triangle AOB = \triangle AOE = \triangle AOF. \qquad (?)$$

$$\therefore AB = AE = AF. \qquad (?)$$

Then, the eight edges terminating at E and F are all equal. (§ 406, I)

Thus, the twelve edges of the octaedron are all equal, and the faces are equal equilateral △s. (?)

Again, by cons., the diagonals of quadrilateral $BEDF$ are equal, and bisect each other at rt. ∠s.

Hence, $BEDF$ is a square equal to $ABCD$, and OA is $\perp$ to its plane. (§ 400)

Then, pyramids A–$BEDF$ and E–$ABCD$ are equal; and hence polyedral ∠s A–$BEDF$ and E–$ABCD$ are equal.

In like manner, any two polyedral ∠s are equal.

Therefore, solid $AEFC$ is a regular octaedron.

4. *To construct a regular dodecaedron.*

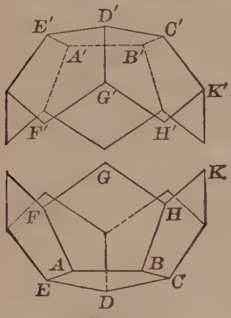


Fig. 1.

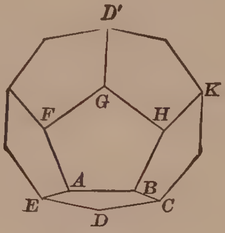


Fig. 2.

Given line AB.

Required to construct with AB as an edge a regular dodecaedron.

Construction. Construct regular pentagon $ABCDE$ (Fig. 1); and to it join five equal regular pentagons, so inclined as to form equal triedral ∠s at A, B, C, D, and E. (§ 460, 1)

Then there is formed a convex surface AK composed of six regular pentagons, as shown in lower part of Fig. 1.

Construct a second surface $A'K'$ equal to AK, as shown in upper part of Fig. 1.

Surfaces AK and $A'K'$ may be combined as shown in Fig. 2, so as to form at F a triedral ∠ equal to that at A, having for its faces the regular pentagons about vertices F and F' in Fig. 1. (§ 460, 1)

Then, solid AK is a regular dodecaedron.

Proof. Since G' falls at G, and diedral ∠ FG and face ∠s FGH and FGD' (Fig. 2) are equal respectively to the diedral ∠ and face ∠s of triedral ∠ F, the faces about vertex G will form a triedral ∠ equal to that at F.

In this way, it may be proved that at each of the vertices H, K, etc., there is formed a triedral ∠ equal to that at F.

Therefore, solid AK is a regular dodecaedron.

5. *To construct a regular icosaedron.*

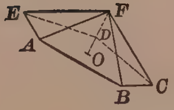


Fig. 1.

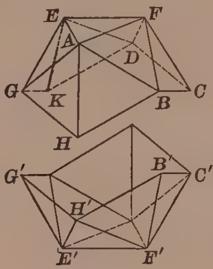


Fig. 2.

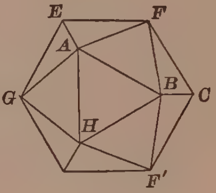


Fig. 3.

Given line *AB*.

Required to construct with *AB* as an edge a regular icosaedron.

Construction. Construct regular pentagon *ABCDE* (Fig. 1); at its centre *O* draw line $OF \perp ABCDE$, making $AF = AB$, and draw lines *AF*, *BF*, *CF*, *DF*, and *EF*.

Then, *F–ABCDE* is a polyedral ∠ composed of five equal equilateral △s. (§§ 406, I, 69)

Then construct two other polyedral ∠s, *A–BFEGH* and *E–AFDKG*, each equal to *F–ABCDE*; and place them as shown in upper part of Fig. 2, so that faces *ABF* and *AEF* of *A–BFEGH*, and faces *AEF* and *DEF* of *E–AFDKG*, shall coincide with the corresponding faces of *F–ABCDE*.

Then there is formed a convex surface *GC*, composed of ten equilateral △s.

Construct a second surface *G'C'* equal to *GC*, as shown in lower part of Fig. 2.

Surfaces *GC* and *G'C'* may be combined as shown in Fig. 3, so that edges *GH* and *HB* shall coincide with edges *G'H'* and *H'B'*, respectively.

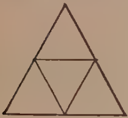
Then, solid *GC* is a regular icosaedron.

Proof. Since diedral ∠s *AH*, *E'H'*, and *F'H'* are equal to the diedral ∠s of polyedral ∠ *F*, the faces about vertices *H* and *H'* form a polyedral ∠ at *H* equal to that at *F*.

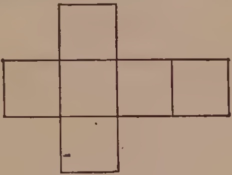
Then, since diedral ∠s FB, AB, HB, and $F'B$ (Fig. 3) are equal to the diedral ∠s of polyedral ∠ F, the faces about vertex B form a polyedral ∠ equal to that at F; and it may be shown that at each of the vertices C, D, etc., there is formed a polyedral ∠ equal to that at F.

Therefore, solid GC is a regular icosaedron.

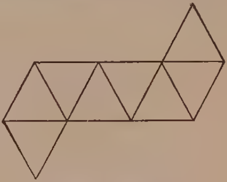
539. Sch. To construct the regular polyedrons, draw the following figures on cardboard; cut them out entire, and on the interior lines cut the cardboard half through; the edges may then be brought together to form the respective solids.



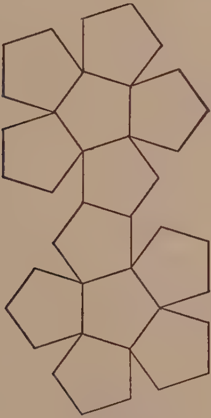
TETRAEDRON.



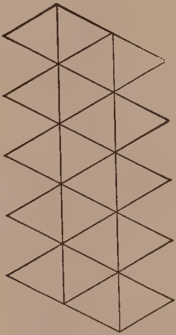
HEXAEDRON.



OCTAEDRON.



DODECAEDRON.



ICOSAEDRON.

EXERCISES.

76. The volume of a pyramid whose altitude is 7 in. is 686 cu. in. Find the volume of a similar pyramid whose altitude is 12 in.

77. If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., find the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.

(Represent the altitude by x.)

78. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second?

79. A pyramid whose altitude is 10 in., weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb.?

80. An edge of a polyedron is 56, and the homologous edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface, and the volume, of the first polyedron.

81. The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume?

(Let x and y denote the homologous edges of the tetraedrons.)

82. The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface?

83. The homologous edges of three similar tetraedrons are 3, 4, and 5, respectively. Find the homologous edge of a similar tetraedron equivalent to their sum.

(Represent the edge by x.)

84. State and prove the converse of Prop. XXVII.

85. The volume of a regular tetraedron is equal to the cube of its edge multiplied by $\frac{1}{12}\sqrt{2}$.

86. The volume of a regular tetraedron is $18\sqrt{2}$. Find the area of its entire surface. (Ex. 85.)

(Represent the edge by x.)

87. The volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

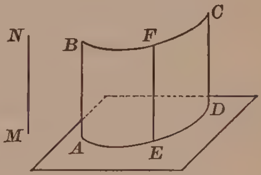
Book VIII.

THE CYLINDER, CONE, AND SPHERE.

DEFINITIONS.

540. A *cylindrical surface* is a surface generated by a moving straight line, which constantly intersects a given plane curve, and in all its positions is parallel to a given straight line, not in the plane of the curve.

Thus, if line AB moves so as to constantly intersect plane curve AD, and is constantly parallel to line MN, not in the plane of the curve, it generates a cylindrical surface.



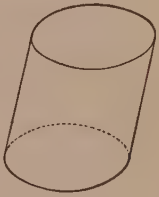
The moving line is called the *generatrix*, and the curve the *directrix*.

Any position of the generatrix, as EF, is called an *element* of the surface.

A *cylinder* is a solid bounded by a cylindrical surface, and two parallel planes.

The parallel planes are called the *bases* of the cylinder, and the cylindrical surface the *lateral surface*.

The *altitude* of a cylinder is the perpendicular distance between the planes of its bases.



A *right cylinder* is a cylinder the elements of whose lateral surface are perpendicular to its bases.

A *circular cylinder* is a cylinder whose base is a circle.

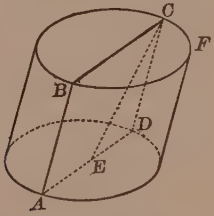
A plane is said to be *tangent* to a cylinder when it contains one, and only one, element of the lateral surface.

541. It follows from the definition of a cylinder (§ 540) that

The elements of the lateral surface of a cylinder are equal and parallel. (§ 415)

Prop. I. Theorem.

542. *A section of a cylinder made by a plane passing through an element of the lateral surface is a parallelogram.*



Given *ABCD* a section of cylinder *AF*, made by a plane passing through *AB*, an element of the lateral surface.

To Prove section *ABCD* a ▱.

Note. It should be observed that, with the above hypothesis, *CD* simply represents the intersection of plane *AC* with the cylindrical surface, and may be a *curved* line; it must be proved that it is a str. line ∥ *AB*.

Proof. *AD* and *BC* are str. lines, and ∥. (§§ 396, 414)

Now draw str. line *CE* in plane *AC* ∥ *AB*; then, *CE* is an element of the cylindrical surface. (§§ 541, 53)

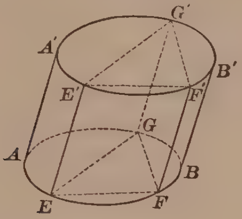
Then since *CE* lies in plane *AC*, and also in the cylindrical surface, it must be the intersection of the plane with the cylindrical surface.

Then, *CD* is a str. line ∥ *AB*, and *ABCD* is a ▱.

543. Cor. *A section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

PROP. II. THEOREM.

544. *The bases of a cylinder are equal.*



Given cylinder AB'.

To Prove base $A'B'$ = base AB.

Proof. Let E', F', and G' be any three points in the perimeter of base $A'B'$, and draw EE', FF', and GG' elements of the lateral surface.

Draw lines EF, FG, GE, $E'F'$, $F'G'$, and $G'E'$.

Now, EE' and FF' are equal and ∥. (§ 541)

Then, $EE'F'F$ is a ▱. (?)

$$\therefore\ E'F' = EF. \quad (?)$$

Similarly, $E'G' = EG$ and $F'G' = FG$.

$$\therefore\ \triangle E'F'G' = \triangle EFG. \quad (?)$$

Then, base $A'B'$ may be superposed upon base AB so that points E', F', and G' shall fall at E, F, and G, respectively.

But E' is any point in the perimeter of $A'B'$.

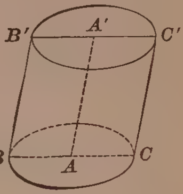
Then, *every* point in the perimeter of $A'B'$ will fall somewhere in the perimeter of AB, and base $A'B'$ = base AB.

545. Cor. I. *The sections of a circular cylinder made by planes parallel to its bases are equal circles.*

For each may be regarded as the upper base of a cylinder whose lower base is a ⊙.

546. Def. The *axis* of a circular cylinder is a straight line drawn between the centres of its bases.

547. Cor. II. *The axis of a circular cylinder is parallel to the elements of its lateral surface.*



Given AA' the axis, and BB' an element of the lateral surface, of circular cylinder BC'.

To Prove $AA' \parallel BB'$.

Proof. Let $BB'C'C$ be a section made by a plane passing through BB' and A; then $BB'C'C$ is a ▱. (§ 542)

$\therefore B'C' = BC$. (?)

Then since BC is a diameter of ⊙ BC, and Ⓢ BC and $B'C'$ are equal, $B'C'$ is a diameter of ⊙ $B'C'$, and passes through A'.

Hence, AB and $A'B'$ are equal and $\parallel$. (?)

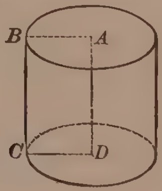
Then, $ABB'A'$ is a ▱. (?)

$\therefore AA' \parallel BB'$.

548. Cor. III. *The axis of a circular cylinder passes through the centres of all sections parallel to the bases.*

PROP. III. THEOREM.

549. *A right circular cylinder may be generated by the revolution of a rectangle about one of its sides as an axis.*



Given rect. $ABCD$.

To Prove the solid generated by the revolution of $ABCD$ about AD as an axis a rt. circular cylinder.

Proof. All positions of BC are $\parallel AD$.

Again, AB and CD generate Ⓢ $\perp AD$. (§ 402)

Then, these Ⓢ are $\parallel$, and $\perp BC$. (§§ 421, 419)

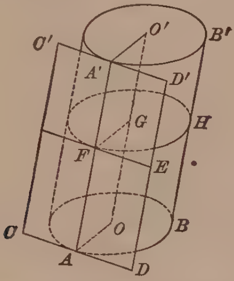
Whence, $ABCD$ generates a rt. circular cylinder.

550. Defs. From the property proved in § 549, a right circular cylinder is called a *cylinder of revolution.*

Similar cylinders of revolution are cylinders generated by the revolution of similar rectangles about homologous sides as axes.

PROP. IV. THEOREM.

551. *A plane drawn through an element of the lateral surface of a circular cylinder and a tangent to the base at its extremity, is tangent to the cylinder.*



Given AA' an element of the lateral surface of circular cylinder AB', line CD tangent to base AB at A, and plane CD' drawn through AA' and CD.

To Prove CD' tangent to the cylinder.

Proof. Let E be any point in plane CD', not in AA', and draw through E a plane $\parallel$ to the bases, intersecting CD' in line EF, and the cylinder in $\odot\, FH$. (§ 545)

Draw axis OO'; then OO' is $\parallel$ AA'. (§ 547)

Let the plane of OO' and AA' intersect the planes of AB and FH in radii OA and GF, respectively. (§ 548)

Then, $GF \parallel OA$ and $FE \parallel AD$. (§ 414)

$\therefore \angle GFE = \angle OAD$. (§ 426)

But $\angle OAD$ is a rt. $\angle$. (§ 170)

Then, FE is $\perp$ GF, and tangent to $\odot\, FH$. (§ 169)

Whence, point E lies without the cylinder.

Then, all portions of CD', not in AA', lie without the cylinder, and CD' is tangent to the cylinder.

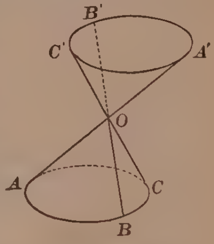
552. Cor. *A plane tangent to a circular cylinder intersects the planes of the bases in lines which are tangent to the bases.*

Ex. 1. The sections of a cylinder made by two parallel planes which cut all the elements of its lateral surface are equal.

THE CONE.

DEFINITIONS.

553. A *conical surface* is a surface generated by a moving straight line, which constantly intersects a given plane curve, and passes through a given point not in the plane of the curve.



Thus, if line OA moves so as to constantly intersect plane curve ABC, and constantly passes through point O, not in the plane of the curve, it generates a conical surface.

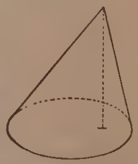
The moving line is called the *generatrix*, and the curve the *directrix.*

The given point is called the *vertex*, and any position of the generatrix, as OB, is called an *element* of the surface.

If the generatrix be supposed indefinite in length, it will generate two conical surfaces of indefinite extent, O–$A'B'C'$ and O–ABC.

These are called the *upper* and *lower nappes*, respectively.

A *cone* is a solid bounded by a conical surface, and a plane cutting all its elements.



The plane is called the *base* of the cone, and the conical surface the *lateral surface.*

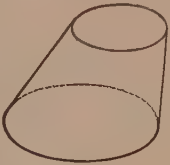
The *altitude* of a cone is the perpendicular distance from the vertex to the plane of the base.

A *circular cone* is a cone whose base is a circle.

The *axis* of a circular cone is a straight line drawn from the vertex to the centre of the base.

A *right circular cone* is a circular cone whose axis is perpendicular to its base.

A *frustum of a cone* is a portion of a cone included between the base and a plane parallel to the base.



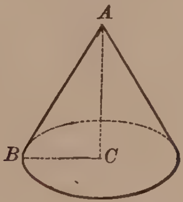
The base of the cone is called the *lower base,* and the section made by the plane the *upper base,* of the frustum.

The *altitude* is the perpendicular distance between the planes of the bases.

A plane is said to be *tangent* to a cone, or frustum of a cone, when it contains one, and only one, element of the lateral surface.

PROP. V. THEOREM.

554. *A right circular cone may be generated by the revolution of a right triangle about one of its legs as an axis.*



Given C the rt. ∠ of rt. △ ABC.

To Prove the solid generated by the revolution of ABC about AC as an axis a right circular cone.

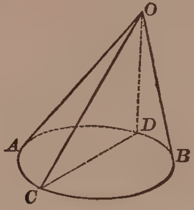
(The proof is left to the pupil.)

555. Defs. From the above property, a right circular cone is called a *cone of revolution.*

Similar cones of revolution are cones generated by the revolution of similar right triangles about homologous legs as axes.

PROP. VI. THEOREM.

556. *A section of a cone made by a plane passing through the vertex is a triangle.*



Given OCD a section of cone OAB made by a plane passing through vertex O.

To Prove section OCD a △.

Proof. We have CD a str. line. (§ 396)

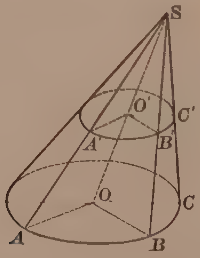
Now draw str. lines in plane OCD from O to C and D; these str. lines are elements of the conical surface. (§ 550)

Then, since these str. lines lie in plane OCD, and also in the conical surface, they must be the intersections of the plane with the conical surface.

Then, OC and OD are str. lines, and OCD is a △.

PROP. VII. THEOREM.

557. *A section of a circular cone made by a plane parallel to the base is a circle.*



Given $A'B'C'$ a section of circular cone S–ABC, made by a plane ∥ to the base.

To Prove $A'B'C'$ a ⊙.

Proof. Draw axis OS, intersecting plane $A'B'C'$ at O'.

Let A' and B' be any two points in perimeter $A'B'C'$.

Let the planes determined by these points and OS intersect the base in radii OA and OB, the section in lines $O'A'$ and $O'B'$, and the lateral surface in lines $SA'A$ and $SB'B$, respectively.

Then, $SA'A$ and $SB'B$ are str. lines. (§ 556)

Now, $O'A' \parallel OA$ and $O'B' \parallel OB$. (§ 414)

Then, $\triangle SO'A'$ and $SO'B'$ are similar to $\triangle SOA$ and SOB, respectively. (§ 257)

$$\therefore \frac{O'A'}{OA} = \frac{SO'}{SO} \text{ and } \frac{O'B'}{OB} = \frac{SO'}{SO}. \qquad (?)$$

$$\therefore \frac{O'A'}{OA} = \frac{O'B'}{OB}. \qquad (?)$$

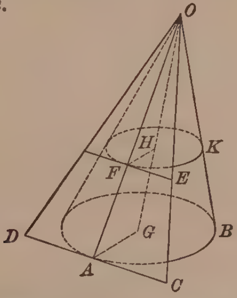
But, $OA = OB$. (§ 143)

Then, $O'A' = O'B'$; and as A' and B' are *any* two points in perimeter $A'B'C'$, section $A'B'C'$ is a ⊙.

558. Cor. *The axis of a circular cone passes through the centre of every section parallel to the base.*

Prop. VIII. Theorem.

559. *A plane drawn through an element of the lateral surface of a circular cone and a tangent to the base at its extremity, is tangent to the cone.*



Given OA an element of the lateral surface of circular cone OAB, line CD tangent to base AB at A, and plane OCD drawn through OA and CD.

To Prove OCD tangent to the cone.

(Prove that E lies without the cone.)

560. Cor. *A plane tangent to a circular cone intersects the plane of the base in a line tangent to the base.*

THE SPHERE.

DEFINITIONS.

561. A *sphere* is a solid bounded by a surface, all points of which are equally distant from a point within called the *centre.*

A *radius* of a sphere is a straight line drawn from the centre to the surface.

A *diameter* is a straight line drawn through the centre, having its extremities in the surface.

562. It follows from the definition of § 561 that *all radii of a sphere are equal.*

Also, all its diameters are equal, since each is the sum of two radii.

563. *Two spheres are equal when their radii are equal.*

For they can evidently be applied one to the other so that their surfaces shall coincide throughout.

Conversely, *the radii of equal spheres are equal.*

564. A line (or a plane) is said to be *tangent to a sphere* when it has one, and only one, point in common with the surface; the common point is called the *point of contact.*

A polyedron is said to be *inscribed in a sphere* when all its vertices lie in the surface of the sphere; in this case the sphere is said to be *circumscribed about the polyedron.*

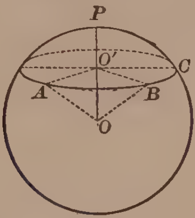
A polyedron is said to be *circumscribed about a sphere* when all its faces are tangent to the sphere; in this case the sphere is said to be *inscribed in the polyedron.*

565. *A sphere may be generated by the revolution of a semicircle about its diameter as an axis.*

For all points of such a surface are equally distant from the centre of the ⊙. (?)

Prop. IX. Theorem.

566. *A section of a sphere made by a plane is a circle.*



Given ABC a section of sphere APC made by a plane.

To Prove ABC a ⊙.

Proof. Let O be the centre of the sphere, and draw line $OO' \perp$ to plane ABC.

Let A and B be any two points in perimeter ABC, and draw lines OA, OB, $O'A$, and $O'B$.

Now, $OA = OB$. (?)

$\therefore O'A = O'B$. (§ 407, I)

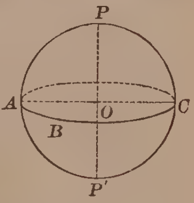
But A and B are *any* two points in perimeter ABC.

Therefore, ABC is a ⊙.

567. Defs. A *great circle* of a sphere is a section made by a plane passing through the centre; as ABC.

A *small circle* is a section made by a plane which does not pass through the centre.

The *diameter* perpendicular to a circle of a sphere is called the *axis* of the circle, and its extremities are called the *poles*.



568. Cor. I. *The axis of a circle of a sphere passes through the centre of the circle.*

569. Cor. II. *All great circles of a sphere are equal.*

For their radii are radii of the sphere.

570. Cor. III. *Every great circle bisects the sphere and its surface.*

For if the portions of the sphere formed by the plane of the great ⊙ be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either surface are equally distant from the centre.

571. Cor. IV. *Any two great circles bisect each other.*

For the intersection of their planes is a diameter of the sphere, and therefore a diameter of each ⊙. (§ 152)

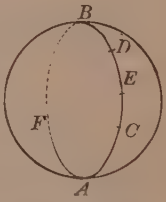
572. Cor. V. *Between any two points on the surface of a sphere, not the extremities of a diameter, an arc of a great circle, less than a semi-circumference, can be drawn, and but one.*

For the two points, with the centre of the sphere, determine a plane which intersects the surface of the sphere in the required arc.

Note. If the points *are* the extremities of a diameter, an indefinitely great number of arcs of great ⓢ can be drawn between them; for an indefinitely great number of planes can be drawn through the diameter.

573. Def. The *distance* between two points on the surface of a sphere, not at the extremities of a diameter, is the arc of a great circle, less than a semi-circumference, drawn between them.

Thus, the distance between points *C* and *D* is arc *CED*, and not arc *CAFBD*.

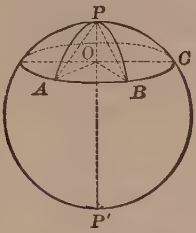


574. Cor. VI. *An arc of a circle may be drawn through any three points on the surface of a sphere.*

For the three points determine a plane which intersects the surface of the sphere in the required arc.

PROP. X. THEOREM.

575. *All points in the circumference of a circle of a sphere are equally distant from each of its poles.*



Given P and P' the poles of ⊙ ABC of sphere APC.

To Prove all points in circumference ABC equally distant (§ 573) from P, and also from P'.

Proof. Let A and B be any two points in circumference ABC, and draw arcs of great Ⓢ PA and PB.

Draw axis PP', intersecting plane ABC at O.

Draw lines OA and OB, and chords PA and PB.

Now O is the centre of ⊙ ABC. (§ 568)

$$\therefore OA = OB. \qquad (?)$$

$$\therefore \text{chord } PA = \text{chord } PB. \qquad (\S\ 406, \text{I})$$

$$\therefore \text{arc } PA = \text{arc } PB. \qquad (\S\ 157)$$

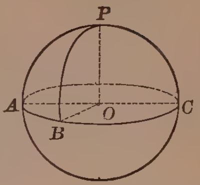
But A and B are *any* two points in circumference ABC.

Therefore, *all* points in circumference ABC are equally distant from P.

In like manner, all points in circumference ABC are equally distant from P'.

576. Def. The *polar distance* of a circle of a sphere is the distance (§ 573) from the nearer of its poles, or from either pole if they are equally near, to the circumference.

Thus, in figure of Prop. X, the polar distance of ⊙ ABC is arc PA.



577. Cor. *All points in the circumference of a great circle of a sphere are at a quadrant's distance from either pole.*

Given P a pole of great ⊙ ABC of sphere APC, B any point in circumference ABC, and PB an arc of a great ⊙.

To Prove arc PB a quadrant (§ 146).

Proof. Let O be the centre of the sphere, and draw radii OB and OP.

Then, $\angle POB$ is a rt. $\angle$. (§ 398)

Whence, arc PB is a quadrant. (§ 191)

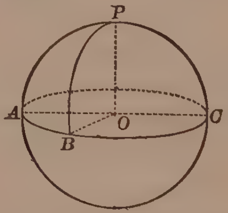
The above proof holds for either pole of the great ⊙.

Note. An arc of a circle may be drawn on the surface of a sphere by placing one foot of the compasses at the nearer pole of the circle, the distance between the feet being equal to the chord of the polar distance.

Prop. XI. Theorem.

578. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points in the arc of a great circle, it is a pole of that arc.*

Note. The term *quadrant*, in Spherical Geometry, usually signifies a quadrant of a great circle.



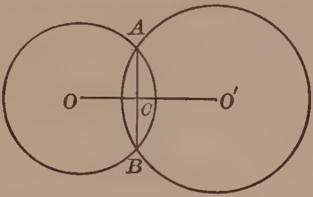
Given point P on surface of sphere APC, AB an arc of great ⊙ ABC, and PA and PB quadrants.

To Prove P a pole of arc AB.

(PO is $\perp$ to OA and OB; then use § 400.)

PROP. XII. THEOREM.

579. *The intersection of two spheres is a circle, whose centre is in the straight line joining the centres of the spheres, and whose plane is perpendicular to that line.*



Given two intersecting spheres.

To Prove their intersection a ⊙, whose centre is in the line joining the centres of the spheres, and whose plane is ⊥ to this line.

Proof. Let O and O' be the centres of two Ⓢ, whose common chord is AB; draw line OO', intersecting AB at C.

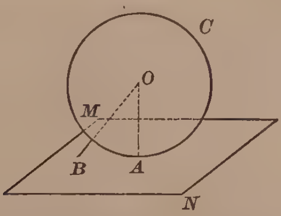
Then, OO' bisects AB at rt. ∡s. (§ 178)

If we revolve the entire figure about OO' as an axis, the Ⓢ will generate spheres whose centres are O and O'. (§ 565)

And AC will generate a ⊙ ⊥ OC', whose centre is C, which is the intersection of the two spheres. (§ 402)

PROP. XIII. THEOREM.

580. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



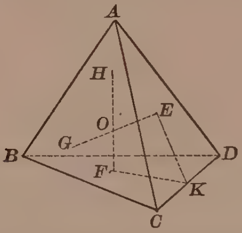
(The proof is left to the pupil; compare § 169.)

581. Cor. (Converse of Prop. XIII.) *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.* (Fig. of Prop. XIII.)

(The proof is left to the pupil; compare § 170.)

PROP. XIV. THEOREM.

582. *Through four points, not in the same plane, a spherical surface can be made to pass, and but one.*



Given A, B, C, and D points not in the same plane.

To Prove that a spherical surface can be passed through A, B, C, and D, and but one.

Proof. Pass planes through A, B, C, and D, forming tetraedron $ABCD$, and let K be the middle point of CD.

Draw lines KE and KF in faces ACD and BCD, respectively, $\perp CD$; and let E and F be the centres of the circumscribed ⓢ of $\triangle$ ACD and BCD, respectively. (§ 222)

Then plane EKF is $\perp CD$. (§ 400)

Draw line $EG \perp ACD$, and line $FH \perp BCD$; then EG and FH lie in plane EKF. (§ 439)

Then EG and FH must meet at some point O, unless they are $\parallel$; this cannot be unless ACD and BCD are in the same plane, which is contrary to the hyp. (§ 418)

Now O, being in EG, is equally distant from A, C, and D; and being in FH, is equally distant from B, C, and D. (§ 406, I)

Then O is equally distant from A, B, C, and D; and a spherical surface described with O as a centre, and OA as a radius, will pass through A, B, C, and D.

Now the centre of any spherical surface passing through A, B, C, and D must be in each of the $\perp$s EG and FH.

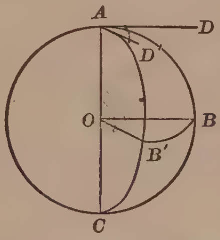
Then as EG and FH intersect in but one point, only one spherical surface can be passed through A, B, C, and D.

583. Defs. *The angle between two intersecting curves* is the angle between tangents to the curves at their point of intersection.

A *spherical angle* is the angle between two intersecting arcs of great circles.

PROP. XV. THEOREM.

584. *A spherical angle is measured by an arc of a great circle having its vertex as a pole, included between its sides produced if necessary.*



Given ABC and $AB'C$ arcs of great ⓢ on the surface of sphere AC, lines AD and AD' tangent to ABC and $A'BC$, respectively, and BB' an arc of a great ⊙ having A as a pole, included between arcs ABC and $AB'C$.

To Prove that $\angle DAD'$ is measured by arc BB'.

Proof. Let O be the centre of the sphere, and draw diameter AOC and lines OB and OB'.

Now, arcs AB and AB' are quadrants. (§ 577)

Whence, $\angle$s AOB and AOB' are rt. $\angle$s. (?)

Therefore, $OB \parallel AD$ and $OB' \parallel AD'$. (§§ 170, 54)

$\therefore \angle DAD' = \angle BOB'$. (§ 426)

But $\angle BOB'$ is measured by arc BB'. (?)

Then, $\angle DAD'$ is measured by arc BB'.

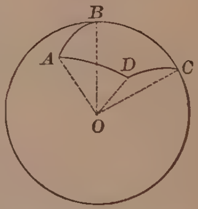
585. Cor. I. (Fig. of Prop. XV.) Plane BOB' is $\perp$ OA. (§ 400)
Then planes ABC and BOB' are $\perp$. (§ 441)
Now a tangent to arc AB at B is $\perp$ BOB'. (§ 439)
Then it is $\perp$ to a tangent to arc BB' at B. (§ 398)
Then, spherical $\angle ABB'$ is a rt. $\angle$. (§ 583)

That is, *an arc of a great circle drawn from the pole of a great circle is perpendicular to its circumference.*

586. Cor. II. *The angle between two arcs of great circles is the plane angle of the diedral angle between their planes.* (§ 429)

SPHERICAL POLYGONS AND SPHERICAL PYRAMIDS.

Definitions.



587. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of *great* circles; as $ABCD$.

The bounding arcs are called the *sides* of the spherical polygon, and are usually measured in *degrees*.

The *angles* of the spherical polygon are the spherical angles (§ 583) between the adjacent sides, and their vertices are called the *vertices* of the spherical polygon.

A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

A *spherical triangle* is a spherical polygon of three sides.

A spherical triangle is called *isosceles* when it has two sides equal; *equilateral* when all its sides are equal; and *right-angled* when it has a right angle.

588. The planes of the sides of a spherical polygon form a polyedral angle, whose vertex is the centre of the sphere, and whose face angles are measured by the sides of the spherical polygon (§ 192).

Thus, in the figure of § 587, the planes of the sides of the spherical polygon form a polyedral angle, *O–ABCD*, whose face ∠s *AOB*, *BOC*, etc., are measured by arcs *AB*, *BC*, etc., respectively.

A spherical polygon is called *convex* when the polyedral angle formed by the planes of its sides is convex (§ 453).

589. A *spherical pyramid* is a solid bounded by a spherical polygon and the planes of its sides; as *O–ABCD*, figure of § 587.

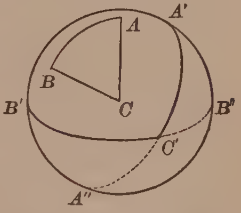
The centre of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon the *base*.

Two spherical pyramids are equal when their bases are equal.

For they can evidently be applied one to the other so as to coincide throughout.

590. If circumferences of great circles be drawn with the vertices of a spherical triangle as poles, they divide the surface of the sphere into eight spherical triangles.

Thus, if circumference *B′C′B″* be drawn with vertex *A* of spherical △ *ABC* as a pole, circumference *A′C″A″* with *B* as a pole, and circumference *A′B″A″B′* with *C* as a pole, the surface of the sphere is divided into eight spherical △s; *A′B′C′*, *A′B″C′*, *A″B′C′*, and *A″B″C′* on the hemisphere represented in the figure, the others on the opposite hemisphere.

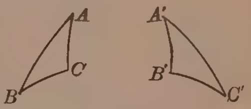


Of these eight spherical △s, one is called the *polar triangle* of *ABC*, and is determined as follows:

Of the intersections, *A′* and *A″*, of circumferences drawn with *B* and *C* as poles, that which is nearer (§ 573) to *A*, *i.e.*, *A′*, is a vertex of the polar triangle; and similarly for the other intersections.

Thus, *A′B′C′* is the polar △ of *ABC*.

591. Two spherical polygons, on the same or equal spheres, are said to be *symmetrical* when the sides and angles of one are equal, respectively, to the sides and angles of the other, if the equal parts occur *in the reverse order.*

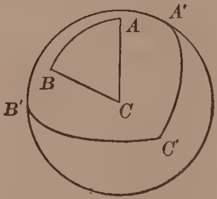


Thus, if spherical △ ABC and $A'B'C'$, on the same or equal spheres, have sides AB, BC, and CA equal, respectively, to sides $A'B'$, $B'C'$, and $C'A'$, and ∠ A, B, and C to ∠ A', B', and C', and the equal parts occur in the reverse order, the △ are symmetrical.

It is evident that, in general, two symmetrical spherical polygons cannot be placed so as to coincide throughout.

Prop. XVI. Theorem.

592. *If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.*



Given $A'B'C'$ the polar △ of spherical △ ABC; A, B, and C being the poles of arcs $B'C'$, $C'A'$, and $A'B'$, respectively.

To Prove ABC the polar △ of spherical △ $A'B'C'$.

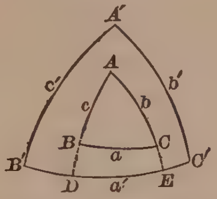
Proof. B is the pole of arc $A'C'$.
Whence, A' lies at a quadrant's distance from B. (§ 577)
Again, C is the pole of arc $A'B'$.
Whence, A' lies at a quadrant's distance from C.
Therefore, A' is the pole of arc BC. (§ 578)
Similarly, B' is the pole of arc CA, and C' of arc AB.
Then, ABC is the polar △ of $A'B'C'$.

For of the two intersections of the circumferences having B' and C', respectively, as poles, A is the nearer to A'; and similarly for the other vertices. (§ 590)

Note. Two spherical triangles, each of which is the polar triangle of the other, are called *polar triangles.*

PROP. XVII. THEOREM.

593. *In two polar triangles, each angle of one is measured by the supplement of that side of the other of which it is the pole.*



Given A, B, C, A', B', and C' the ∠s, expressed in degrees, of polar △s ABC and $A'B'C'$; A being the pole of $B'C'$, B of $C'A'$, C of $A'B'$, A' of BC, B' of CA, and C' of AB.

Let sides BC, CA, AB, $B'C'$, $C'A'$, and $A'B'$, expressed in degrees, be denoted by a, b, c, a', b', and c', respectively.

To Prove

$$A = 180° - a', \qquad B = 180° - b', \qquad C = 180° - c',$$
$$A' = 180° - a, \qquad B' = 180° - b, \qquad C' = 180° - c.$$

Proof. Produce arcs AB and AC to meet arc $B'C'$ at D and E, respectively.

Since B' is the pole of arc AE, and C' of arc AD, arcs $B'E$ and $C'D$ are quadrants. (§ 577)

$$\therefore \text{ arc } B'E + \text{arc } C'D = 180°.$$

Or,
$$\text{arc } DE + \text{arc } B'C' = 180°.$$

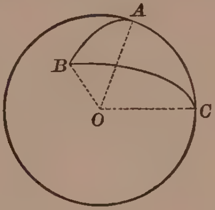
But since A is the pole of arc $B'C'$, arc DE is the measure of ∠ A. (§ 584)

$$\therefore A + a' = 180°, \text{ or } A = 180° - a'.$$

In like manner, the theorem may be proved for any ∠ of either △.

PROP. XVIII. THEOREM.

594. *Any side of a spherical triangle is less than the sum of the other two sides.*



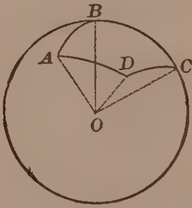
Given AB any side of spherical $\triangle ABC$.

To Prove $AB < AC + BC$.

(By § 457, $\angle AOB < \angle AOC + \angle BOC$; and these $\angle$s are measured by sides AB, AC, and BC, respectively.)

PROP. XIX. THEOREM.

595. *The sum of the sides of a convex spherical polygon is less than* 360°.



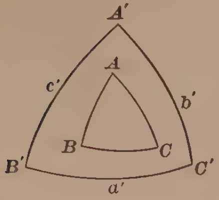
Given convex spherical polygon $ABCD$.

To Prove $AB + BC + CD + DA < 360°$.

(By § 458, sum of $\angle$s AOB, BOC, COD, and DOA is $< 360°$.)

PROP. XX. THEOREM.

596. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Given A, B, and C the ∡s, expressed in degrees, of spherical △ ABC.

To Prove $A + B + C > 180°$, and $< 540°$.

Proof. Let $A'B'C'$ be the polar △ of spherical △ ABC, A being the pole of $B'C'$, B of $C'A'$, and C of $A'B'$.

Also, let sides $B'C'$, $C'A'$, and $A'B'$, expressed in degrees, be denoted by a', b', and c', respectively.

Then,
$$A = 180° - a',$$
$$B = 180° - b',$$
$$C = 180° - c'. \qquad (\S\ 593)$$

Adding these equations, we have
$$A + B + C = 540° - (a' + b' + c'). \qquad (1)$$
$$\therefore\ A + B + C < 540°.$$

Again, $a' + b' + c' < 360°$. (§ 595)

Whence, by (1), $A + B + C > 180°$.

597. Cor. *A spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.*

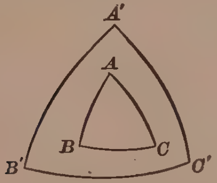
DEFINITIONS.

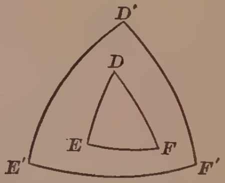
598. A spherical triangle having two right angles is called a *bi-rectangular triangle*, and one having three right angles a *tri-rectangular triangle.*

599. Two spherical polygons on the same sphere, or equal spheres, are said to be *mutually equilateral*, or *mutually equiangular*, when the sides or angles of one are equal, respectively, to the homologous sides or angles of the other, *whether taken in the same or in the reverse order.*

PROP. XXI. THEOREM.

600. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, their polar triangles are mutually equilateral.*





Given ABC and DEF mutually equiangular spherical $\triangle$s on the same sphere, or equal spheres, $\angle$s A and D being homologous; also, $A'B'C'$ the polar $\triangle$ of ABC, and $D'E'F'$ of DEF, A being the pole of $B'C'$, and D of $E'F'$.

To Prove $A'B'C'$ and $D'E'F'$ mutually equilateral.

Proof. $\angle$s A and D are measured by the supplements of sides $B'C'$ and $E'F'$, respectively. (§ 593)

But by hyp., $\angle A = \angle D$.

$\therefore B'C' = E'F'$. (§ 31, 2)

In like manner, any two homologous sides of $A'B'C'$ and $D'E'F'$ may be proved equal.

Then, $A'B'C'$ and $D'E'F'$ are mutually equilateral.

601. Cor. (Converse of Prop. XXI.) *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.*

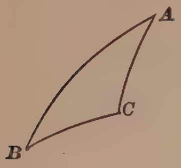
(The proof is left to the pupil; compare § 600.)

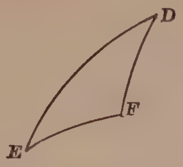
PROP. XXII. THEOREM.

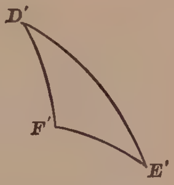
602. *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal, respectively, to two sides and the included angle of the other,*

I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*







I. **Given** ABC and DEF spherical ⟁ on the same sphere, or equal spheres, having

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D;$$

the equal parts occurring in the same order.

To Prove $\triangle ABC = \triangle DEF.$

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that $\angle A$ shall coincide with its equal $\angle D$; side AB falling on side DE, and side AC on side DF.

Then, since $AB = DE$ and $AC = DF$, point B will fall on point E, and point C on point F.

Whence, arc BC will coincide with arc EF. (§ 572)

Hence, ABC and DEF coincide throughout, and are equal.

II. **Given** ABC and $D'E'F'$ spherical ⟁ on the same sphere, or equal spheres, having

$$AB = D'E',\ AC = D'F', \text{ and } \angle A = \angle D';$$

the equal parts occurring in the reverse order.

To Prove ABC and $D'E'F'$ symmetrical.

Proof. Let DEF be a spherical $\triangle$ on the same sphere, or an equal sphere, symmetrical to $D'E'F'$, having

$$DE = D'E',\ DF = D'F', \text{ and } \angle D = \angle D';$$

the equal parts occurring in the reverse order.

Then, in spherical ⟁ ABC and DEF, we have

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D;$$

and the equal parts occur in the same order. (Ax. 1)

$\therefore \triangle ABC = \triangle DEF.$ (§ 602, I)

Therefore, $\triangle ABC$ is symmetrical to $\triangle D'E'F'$.

PROP. XXIII. THEOREM.

603. *If two spherical triangles on the same sphere, or equal spheres, have a side and two adjacent angles of one equal, respectively, to a side and two adjacent angles of the other,*

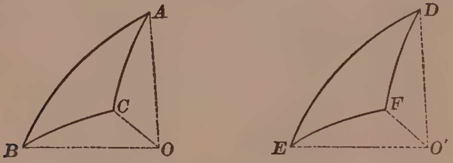
I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*

(The proof is left to the pupil; compare § 602.)

PROP. XXIV. THEOREM.

604. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, they are mutually equiangular.*



Given ABC and DEF mutually equilateral spherical $\triangle$ on equal spheres; sides BC and EF being homologous.

To Prove ABC and DEF mutually equiangular.

Proof. Let O and O' be the centres of the respective spheres, and draw lines OA, OB, OC, $O'D$, $O'E$, and $O'F$.

Now the triedral $\angle$s $O\text{–}ABC$ and $O'\text{–}DEF$ have their homologous face $\angle$s equal. (§ 192)

$\therefore$ diedral $\angle\, OA =$ diedral $\angle\, O'D$. (§ 459)

But the $\angle$ between arcs AB and AC is the plane $\angle$ of diedral $\angle\, OA$, and the $\angle$ between arcs DE and DF is the plane $\angle$ of diedral $\angle\, O'D$. (§ 586)

$\therefore \angle BAC = \angle EDF$. (§ 434)

In like manner, any two homologous $\angle$s of ABC and DEF may be proved equal.

Whence, ABC and DEF are mutually equiangular.

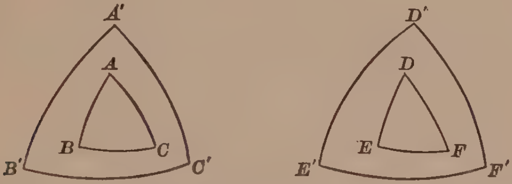
Note. The theorem may be proved in a similar manner when the given spherical ⩓ are on the same sphere.

605. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

PROP. XXV. THEOREM.

606. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, they are mutually equilateral.*



Given ABC and DEF mutually equiangular spherical ⩓ on the same sphere, or equal spheres.

To Prove ABC and DEF mutually equilateral.

Proof. Let $A'B'C'$ be the polar △ of ABC, and $D'E'F'$ of DEF.

Then, since ABC and DEF are mutually equiangular, $A'B'C'$ and $D'E'F'$ are mutually equilateral. (§ 600)

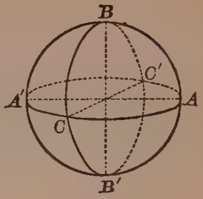
Then $A'B'C'$ and $D'E'F'$ are mutually equiangular. (§ 604)

But ABC is the polar △ of $A'B'C'$, and DEF of $D'E'F'$. (§ 592)

Then ABC and DEF are mutually equilateral. (§ 600)

607. Cor. I. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*



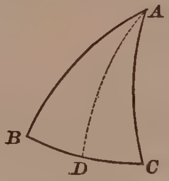
608. Cor. II. *If three diameters of a sphere be so drawn that each is perpendicular to the other two, the planes determined by them divide the surface of the sphere into eight equal tri-rectangular triangles.*

(Prove by § 607, 1. By § 585, each ∠ of each spherical △ is a rt. ∠.)

609. Cor. III. *The surface of a sphere is eight times the surface of one of its tri-rectangular triangles.*

Prop. XXVI. Theorem.

610. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*



Given, in spherical △ ABC, $AB = AC$.

To Prove $\angle B = \angle C$.

Proof. Draw AD an arc of a great ⊙, bisecting side BC at D.

In spherical ⚠ ABD and ACD, $AD = AD$.

Also, $AB = AC$ and $BD = CD$.

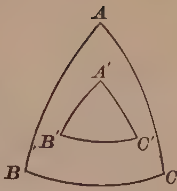
Then, ABD and ACD are mutually equiangular. (§ 604)

$$\therefore \angle B = \angle C.$$

611. Cor. I. *An isosceles spherical triangle is equal to the spherical triangle which is symmetrical to it.*

For the equal parts occur in the same order.

612. Cor. II. (Converse of Prop. XXVI.) *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Given, in spherical △ ABC, ∠ $B =$ ∠ C.

To Prove $AB = AC$.

Proof. Let $A'B'C'$ be the polar △ of ABC; B being the pole of $A'C'$, and C of $A'B'$.

Then, $A'C'$ is the sup. of ∠ B, and $A'B'$ of ∠ C. (§ 593)

∴ $A'C' = A'B'$. (§ 31, 2)

∴ ∠ $B' =$ ∠ C'. (§ 610)

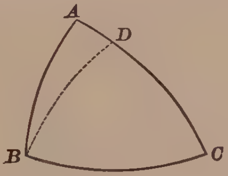
But ABC is the polar △ of $A'B'C'$; B' being the pole of AC, and C' of AB. (§ 592)

Then AB is the sup. of ∠ C', and AC of ∠ B'. (?)

∴ $AB = AC$. (?)

Prop. XXVII. Theorem.

613. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Given, in spherical △ ABC, ∠ $ABC >$ ∠ C.

To Prove $AC > AB$.

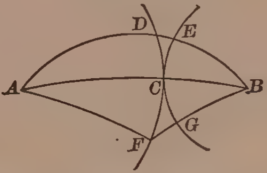
(Prove by a method analogous to that of § 99. Draw BD an arc of a great ⊙ meeting AC at D, and making ∠ CBD equal to ∠ C.)

614. Cor. (Converse of Prop. XXVII.) *If two sides of a spherical triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*

(Prove by *Reductio ad Absurdum.*)

PROP. XXVIII. THEOREM.

615. *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semi-circumference, which joins the points.*



Given points A and B on the surface of a sphere, and AB an arc of a great ⊙, not > a semi-circumference.

To Prove AB the shortest line on the surface of the sphere between A and B.

Proof. Let C be any point in arc AB.

Let DCF and ECG be arcs of small ⓢ with A and B, respectively, as poles, and AC and BC as polar distances.

Now arcs DCF and ECG have only point C in common.

For let F be any other point in arc DCF, and draw arcs of great ⓢ AF and BF.

$$\therefore AF = AC. \qquad (\S\ 575)$$

But, $$AF + BF > AC + BC. \qquad (\S\ 594)$$

Subtracting arc AF from the first member of the inequality, and its equal arc AC from the second member,

$$BF > BC, \text{ or } BF > BG. \qquad (\S\ 575)$$

Whence, F lies without small ⊙ ECG, and arcs DCF and ECG have only point C in common.

We will next prove that the shortest line on the surface of the sphere from A to B must pass through C.

Let $ADEB$ be any line on the surface of the sphere between A and B, not passing through C, and cutting arcs DCF and ECG at D and E, respectively.

Then, whatever the nature of line AD, it is evident that an equal line can be drawn from A to C.

In like manner, whatever the nature of line *BE*, an equal line can be drawn from *B* to *C*.

Hence, a line can be drawn from *A* to *B* passing through *C*, equal to the sum of lines *AD* and *BE*, and consequently < line *ADEB* by the portion *DE*.

Therefore, no line which does not pass through *C* can be the shortest line between *A* and *B*.

But by hyp., *C* is *any* point in arc *AB*.

Hence, the shortest line from *A* to *B* must pass through *every* point of *AB*.

Then, the arc of a great ⊙ *AB* is the shortest line on the surface of the sphere between *A* and *B*.

EXERCISES.

2. If the sides of a spherical triangle are 77°, 123°, and 95°, how many degrees are there in each angle of its polar triangle?

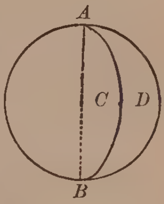
3. If the angles of a spherical triangle are 86°, 131°, and 68°, how many degrees are there in each side of its polar triangle?

MEASUREMENT OF SPHERICAL POLYGONS.

DEFINITIONS.

616. A *lune* is a portion of the surface of a sphere bounded by two semi-circumferences of great circles; as *ACBD*.

The *angle* of the lune is the angle between its bounding arcs.



617. A *spherical wedge* is a solid bounded by a lune and the planes of its bounding arcs.

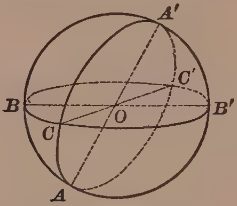
The lune is called the *base* of the spherical wedge.

618. It is evident that *two lunes on the same sphere, or equal spheres, are equal when their angles are equal.*

619. It is evident that *two spherical wedges in the same sphere, or equal spheres, are equal when the angles of the lunes which form their bases are equal.*

PROP. XXIX. THEOREM.

620. *The spherical triangles corresponding to a pair of vertical triedral angles are symmetrical.*



Given AOA', BOB', and COC' diameters of sphere AC; also, the planes determined by them, intersecting the surface in circumferences $ABA'B'$, $BCB'C'$, and $CAC'A'$.

To Prove spherical $\triangle$ ABC and $A'B'C'$ symmetrical.

Proof. $\angle$ AOB, BOC, and COA are equal, respectively, to $\angle$ $A'OB'$, $B'OC'$, and $C'OA'$. (§ 40)

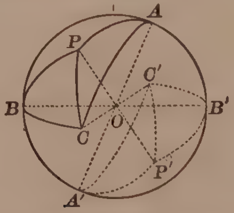
Then, $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. (§ 192)

But the equal parts of ABC and $A'B'C'$ occur *in the reverse order.*

Whence, ABC and $A'B'C'$ are symmetrical. (§ 605, 2)

PROP. XXX. THEOREM.

621. *Two spherical triangles corresponding to a pair of vertical triedral angles are equivalent.*



Given AOA', BOB', and COC' diameters of sphere AB; also, the planes determined by them, intersecting the surface in arcs AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

To Prove area $ABC =$ area $A'B'C'$.

Proof. Let P be the pole of the small ⊙ passing through points A, B, and C, and draw arcs of great Ⓢ PA, PB, and PC.

$$\therefore\ PA = PB = PC. \qquad (\S\ 575)$$

Draw the diameter of the sphere PP', and the arcs of great Ⓢ $P'A'$, $P'B'$, and $P'C'$; then, spherical $\triangle$ PAB and $P'A'B'$ are symmetrical. (§ 620)

But spherical $\triangle\, PAB$ is isosceles.

$$\therefore\ \triangle\, PAB = \triangle\, P'A'B'. \qquad (\S\ 611)$$

In like manner,

$$\triangle\, PBC = \triangle\, P'B'C' \text{ and } \triangle\, PCA = \triangle\, P'C'A'.$$

Then the sum of the areas of $\triangle$ PAB, PBC, and PCA equals the sum of the areas of $P'A'B'$, $P'B'C'$, and $P'C'A'$.

$$\therefore\ \text{area } ABC = \text{area } A'B'C'.$$

622. Sch. If P and P' fall without spherical $\triangle$ ABC and $A'B'C'$, we should take the sum of the areas of two isosceles spherical $\triangle$, diminished by the area of a third.

623. Cor. I. *Two symmetrical spherical triangles are equivalent.*

624. Cor. II. Spherical pyramids $O\text{–}APB$, $O\text{–}BPC$, and $O\text{–}CPA$ are equal, respectively, to spherical pyramids $O\text{–}A'P'B'$, $O\text{–}B'P'C'$, and $O\text{–}C'P'A'$. (§ 589)

$$\therefore\ \text{vol. } O\text{–}ABC = \text{vol. } O\text{–}A'B'C'.$$

Whence, *the spherical pyramids corresponding to a pair of vertical triedral angles are equivalent.*

EXERCISES.

4. The sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles. (§ 596.)

5. The sum of the angles of a spherical polygon of n sides is greater than $2n - 4$, and less than $2n$, right angles.

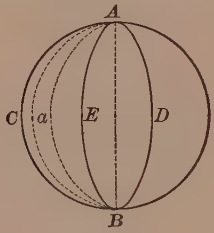
6. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

PROP. XXXI. THEOREM.

625. *Two lunes on the same sphere, or equal spheres, are to each other as their angles.*

Note. The word "*lune*," in the above statement, signifies the *area* of the lune.

Case I. *When the angles are commensurable.*



Given $ACBD$ and $ACBE$ lunes on sphere AB, having their $\angle s$ CAD and CAE commensurable.

To Prove $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

Proof. Let $\angle CAa$ be a common measure of $\angle s$ CAD and CAE, and let it be contained 5 times in $\angle CAD$, and 3 times in $\angle CAE$.

$$\therefore \frac{\angle CAD}{\angle CAE} = \frac{5}{3}. \qquad (1)$$

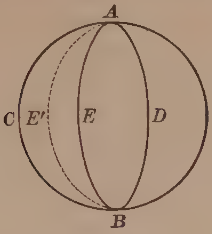
Producing the arcs of division of $\angle CAD$ to B, lune $ACBD$ will be divided into 5 parts, and lune $ACBE$ into 3 parts, all of which parts will be equal. (§ 618)

$$\therefore \frac{ACBD}{ACBE} = \frac{5}{3}. \qquad (2)$$

From (1) and (2), $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}. \qquad (?)$$

Note. The theorem may be proved in a similar manner when the given lunes are on equal spheres.

Case II. *When the angles are incommensurable.*



(Prove as in §§ 189 or 244. Let ∠ CAD be divided into any number of equal parts, and apply one of these parts to ∠ CAE as a unit of measure.)

626. Cor. I. *The surface of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

For the surface of a sphere may be regarded as a lune whose ∠ is equal to 4 rt. ∠s.

627. Cor. II. *If the unit of measure for angles is the right angle, the area of a lune is equal to twice its angle, multiplied by the area of a tri-rectangular triangle.*

Given L the area of a lune; A the numerical measure of its ∠ referred to a rt. ∠ as the unit of measure; and T the area of a tri-rectangular △.

To Prove $L = 2A \times T$.

Proof. The area of the surface of the sphere is $8T$. (§ 609)

$$\therefore \frac{L}{8T} = \frac{A}{4}. \qquad (§ 625)$$

$$\therefore L = \frac{A}{4} \times 8T = 2A \times T.$$

628. Sch. I. Let it be required to find the area of a lune whose ∠ is 50°, on a sphere the area of whose surface is 72.

The ∠ of the lune referred to a rt. ∠ as the unit of measure is $\frac{5}{9}$; and T is $\frac{1}{8}$ of 72, or 9.

Then the area of the lune is $2 \times \frac{5}{9} \times 9$, or 10.

629. Def. A *tri-rectangular pyramid* is a spherical pyramid whose base is a tri-rectangular triangle.

630. Sch. II. It may be proved, as in § 625, that

Two spherical wedges in the same sphere, or equal spheres, are to each other as the angles of the lunes which form their bases.

(The proof is left to the pupil; see § 619.)

631. Sch. III. It may be proved that

If the unit of measure for angles is the right angle, the volume of a spherical wedge is equal to twice the angle of the lune which forms its base, multiplied by the volume of a tri-rectangular pyramid.

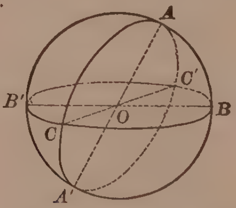
(The proof is left to the pupil; see §§ 626 and 627.)

632. Def. The *spherical excess* of a spherical triangle is the excess of the sum of its angles above 180° (§ 596).

Thus, if the ⦣ of a spherical △ are 65°, 80°, and 95°, its spherical excess is $65° + 80° + 95° - 180°$, or 60°.

Prop. XXXII. Theorem.

633. *If the unit of measure for angles is the right angle, the area of a spherical triangle is equal to its spherical excess, multiplied by the area of a tri-rectangular triangle.*



Given A, B, and C the numerical measures of the ⦣ of spherical △ ABC, referred to a rt. ∠ as the unit of measure, and T the area of a tri-rectangular △.

To Prove area $ABC = (A + B + C - 2) \times T$.

Proof. Complete circumferences $ABA'B'$, $ACA'C'$, and $BCB'C'$, and draw diameters AA', BB', and CC'.

Then, since $ABA'C$ is a lune whose ∠ is A, we have

$$\text{area } ABC + \text{area } A'BC = 2\,A \times T \text{ (§ 626).} \qquad (1)$$

And since $BAB'C$ is a lune whose ∠ is B,

$$\text{area } ABC + \text{area } AB'C = 2\,B \times T. \qquad (2)$$

Again, $\text{area } A'B'C = \text{area } ABC'$. (§ 620)

Adding area ABC to both members, we have

$$\begin{aligned}\text{area } ABC + \text{area } A'B'C &= \text{area of lune } CBC'A \\ &= 2\,C \times T. \qquad (3)\end{aligned}$$

Adding (1), (2), and (3), and observing that the sum of the areas of ⨺ ABC, $A'BC$, $AB'C$, and $A'B'C$ is equal to the area of the surface of a hemisphere, or $4\,T$, we have

$$2 \text{ area } ABC + 4\,T = (2\,A + 2\,B + 2\,C) \times T.$$

$$\therefore \text{ area } ABC + 2\,T = (A + B + C) \times T.$$

$$\therefore \text{ area } ABC = (A + B + C - 2) \times T.$$

634. Sch. I. Let it be required to find the area of a spherical △ whose ∠s are 105°, 80°, and 95°, on a sphere the area of whose surface is 144.

The spherical excess of the spherical △ is 100°, or $\frac{10}{9}$ referred to a rt. ∠ as the unit of measure; and the area of a tri-rectangular △ is $\frac{1}{8}$ of 144, or 18.

Then the area of the spherical △ is $\frac{10}{9} \times 18$, or 20.

635. Sch. II. It may be proved, as in § 633, that

If the unit of measure for angles is the right angle, the volume of a triangular spherical pyramid is equal to the spherical excess of its base, multiplied by the volume of a tri-rectangular pyramid.

(The proof is left to the pupil; see §§ 624 and 630.)

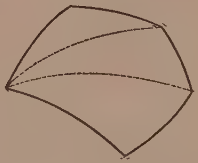
EXERCISES.

7. What is the volume of a spherical wedge the angle of whose base is 127° 30′, if the volume of the sphere is 112?

8. In figure of Prop. XVII., prove $A' = 180° - a$.

PROP. XXXIII. THEOREM.

636. *If the unit of measure for angles is the right angle, the area of any spherical polygon is equal to the sum of its angles, diminished by as many times two right angles as the figure has sides less two, multiplied by the area of a tri-rectangular triangle.*



Given K the area of any spherical polygon, n the number of its sides, s the sum of its ∠s referred to a rt. ∠ as the unit of measure, and T the area of a tri-rectangular △.

To Prove $K = [s - 2(n - 2)] \times T$.

Proof. The spherical polygon can be divided into $n - 2$ spherical △s by drawing diagonals from any vertex.

Now, if the unit of measure for ∠s is the rt. ∠, the area of each spherical △ is equal to the sum of its ∠s, less 2 rt. ∠s, multiplied by T. (§ 633)

Hence, if the unit of measure for ∠s is the rt. ∠, the sum of the areas of the spherical △s is equal to the sum of their ∠s, diminished by $n - 2$ times 2 rt. ∠s, multiplied by T.

But the sum of the ∠s of the spherical △s is equal to the sum of the ∠s of the spherical polygon.

Whence, $K = [s - 2(n - 2)] \times T$.

637. Sch. It may be proved, as in § 636, that

If the unit of measure for angles is the right angle, the volume of any spherical pyramid is equal to the sum of the angles of its base, diminished by as many times two right angles as the base has sides less two, multiplied by the volume of a tri-rectangular pyramid.

(The proof is left to the pupil.)

EXERCISES.

9. The area of a lune is $28\frac{4}{5}$. If the area of the surface of the sphere is 120, what is the angle of the lune?

10. Find the area of a spherical triangle whose angles are 103°, 112°, and 127°, on a sphere the area of whose surface is 160.

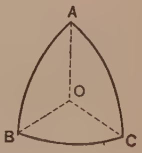
11. Find the volume of a triangular spherical pyramid the angles of whose base are 92°, 119°, and 134°; the volume of the sphere being 192.

12. What is the ratio of the areas of two spherical triangles on the same sphere whose angles are 94°, 135°, and 146°, and 87°, 105°, and 118°, respectively?

13. The area of a spherical triangle, two of whose angles are 78° and 99°, is $34\frac{1}{8}$. If the area of the surface of the sphere is 234, what is the other angle?

14. The volume of a triangular spherical pyramid, the angles of whose base are 105°, 126°, and 147°, is $60\frac{1}{2}$; what is the volume of the sphere?

15. The sides opposite the equal angles of a bi-rectangular triangle are quadrants. (§ 442.)



16. The sides of a spherical triangle, on a sphere the area of whose surface is 156, are 44°, 63°, and 97°. Find the area of its polar triangle.

17. Find the area of a spherical hexagon whose angles are 120°, 139°, 148°, 155°, 162°, and 167°, on a sphere the area of whose surface is 280.

18. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109°, 128°, 137°, 153°, and 158°; the volume of the sphere being 180.

19. The volume of a quadrangular spherical pyramid, the angles of whose base are 110°, 122°, 135°, and 146°, is $12\frac{3}{4}$; what is the volume of the sphere?

20. The area of a spherical pentagon, four of whose angles are 112°, 131°, 138°, and 168°, is 27. If the area of the surface of the sphere is 120, what is the other angle?

21. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere. (§ 400.)

22. The sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of any side, is less than the sum of the other two sides of the triangle.

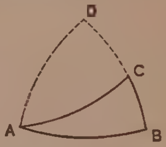
(Compare § 48.)

23. How many degrees are there in the polar distance of a circle, whose plane is $5\sqrt{2}$ units from the centre of the sphere, the diameter of the sphere being 20 units?

(The radius of the ⊙ is a leg of a rt. △, whose hypotenuse is the radius of the sphere, and whose other leg is the distance from its centre to the plane of the ⊙.)

24. The chord of the polar distance of a circle of a sphere is 6. If the radius of the sphere is 5, what is the radius of the circle?

25. If side AB of spherical triangle ABC is a quadrant, and side BC less than a quadrant, prove $\angle A$ less than 90°.

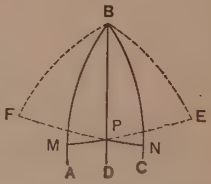


26. The polar distance of a circle of a sphere is 60°. If the diameter of the circle is 6, find the diameter of the sphere, and the distance of the circle from its centre.

(Represent radius of sphere by $2x$.)

27. Any point in the arc of a great circle bisecting a spherical angle is equally distant (§ 573) from the sides of the angle.

(To prove arc $PM =$ arc PN. Let E be a pole of arc AB, and F of arc BC. Spherical $\triangle$ BPE and BPF are symmetrical by § 602, II., and $PE = PF$.)



28. A point on the surface of a sphere, equally distant from the sides of a spherical angle, lies in the arc of a great circle bisecting the angle.

(Fig. of Ex. 27. To prove $\angle ABP = \angle CBP$. Spherical $\triangle$ BPE and BPF are symmetrical by § 605, 2.)

29. The arcs of great circles bisecting the angles of a spherical triangle meet in a point equally distant from the sides of the triangle. (Exs. 27, 28, p. 358.)

30. A circle may be inscribed in any spherical triangle.

31. State and prove the theorem for spherical triangles analogous to Prop. IX., I., Book I.

32. State and prove the theorem for spherical triangles analogous to Prop. V., Book I.

33. State and prove the theorem for spherical triangles analogous to Prop. L., Book I. (Ex. 32.)

34. If PA, PB, and PC are three equal arcs of great circles drawn from point P to the circumference of great circle ABC, prove P a pole of ABC.

(PA and PB are quadrants by Ex. 15, p. 357.)

35. The spherical polygons corresponding to a pair of vertical polyedral angles are symmetrical. (§ 456.)

36. A sphere may be inscribed in, or circumscribed about, any tetraedron. (Ex. 73, Book VII.)

37. What is the locus of points in space at a given distance from a given straight line?

38. Equal small circles of a sphere are equally distant from the centre.

39. State and prove the converse of Ex. 38.

40. The less of two small circles of a sphere is at the greater distance from the centre.

41. State and prove the converse of Ex. 40.

42. What is the locus of points on the surface of a sphere equally distant from the sides of a spherical angle?

43. If two spheres are tangent to the same plane at the same point, the straight line joining their centres passes through the point of contact.

44. The distance between the centres of two spheres whose radii are 25 and 17, respectively, is 28. Find the diameter of their circle of intersection, and its distance from the centre of each sphere.

45. If a polyedron be circumscribed about each of two equal spheres, the volumes of the polyedrons are to each other as the areas of their surfaces.

(Find the volume of each polyedron by dividing it into pyramids.)

46. Either angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

(Fig. of Prop. XX. To prove $\angle A > 180° - (\angle B + \angle C)$, or $> (\angle B + \angle C) - 180°$, according as $\angle B + \angle C$ is $<$ or $>$ 180°. In the latter case, $A'C' + A'B' > B'C'$; then use § 593.)

Book IX.

MEASUREMENT OF THE CYLINDER, CONE, AND SPHERE.

THE CYLINDER.

Definitions.

638. The *lateral area* of a cylinder is the area of its lateral surface.

A *right section* of a cylinder is a section made by a plane perpendicular to the elements of its lateral surface.

639. A prism is said to be *inscribed in a cylinder* when its lateral edges are elements of the cylindrical surface.

In this case, the bases of the prism are inscribed in the bases of the cylinder.

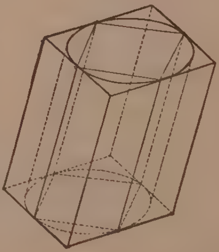
A prism is said to be *circumscribed about a cylinder* when its lateral faces are tangent to the cylinder, and its bases lie in the same planes with the bases of the cylinder.

In this case, the bases of the prism are circumscribed about the bases of the cylinder.

640. It follows from § 363 that

If a prism whose base is a regular polygon be inscribed in, or circumscribed about, a circular cylinder (§ 540), *and the number of its faces be indefinitely increased,*

1. *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*

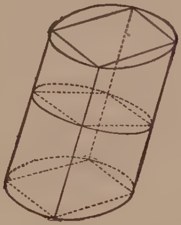


2. *The volume of the prism approaches the volume of the cylinder as a limit.*

3. *The perimeter of a right section of the prism approaches the perimeter of a right section of the cylinder as a limit.**

PROP. I. THEOREM.

641. *The lateral area of a circular cylinder is equal to the perimeter of a right section multiplied by an element of the lateral surface.*



Given S the lateral area, P the perimeter of a rt. section, and E an element of the lateral surface, of a circular cylinder.

To Prove $S = P \times E$.

Proof. Inscribe in the cylinder a prism whose base is a regular polygon, and let S' denote its lateral area, and P' the perimeter of a rt. section.

Then, since the lateral edge of the prism is E,

$$S' = P' \times E. \qquad (\S\ 484)$$

Now let the number of faces of the prism be indefinitely increased.

Then, S' approaches the limit S,

and $P' \times E$ approaches the limit $P \times E$. (§ 640, 1, 3)

By the Theorem of Limits, these limits are equal. (§ 188)

$$\therefore\ S = P \times E.$$

---

* For rigorous proofs of these statements, see Appendix, p. 386.

642. Cor. I. *The lateral area of a cylinder of revolution is equal to the circumference of its base multiplied by its altitude.*

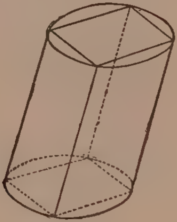
643. Cor. II. If S denotes the lateral area, T the total area, H the altitude, and R the radius of the base, of a cylinder of revolution,

$$S = 2\pi RH. \qquad (\S\ 368)$$

And, $T = 2\pi RH + 2\pi R^2$ (§ 371) $= 2\pi R(H + R)$.

PROP. II. THEOREM.

644. *The volume of a circular cylinder is equal to the product of its base and altitude.*



Given V the volume, B the area of the base, and H the altitude, of a circular cylinder.

To Prove $V = B \times H$.

Proof. Inscribe in the cylinder a prism whose base is a regular polygon, and let V' denote its volume, and B' the area of its base.

Then, since the altitude of the prism is H,

$$V' = B' \times H. \qquad (\S\ 499)$$

Now let the number of faces of the prism be indefinitely increased.

Then, V' approaches the limit V. (§ 640, 2)

And, $B' \times H$ approaches the limit $B \times H$. (§ 363, II)

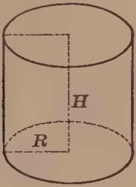
$$\therefore\ V = B \times H. \qquad (?)$$

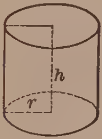
645. Cor. If V denotes the volume, H the altitude, and R the radius of the base, of a circular cylinder,

$$V = \pi R^2 H. \qquad (?)$$

PROP. III. THEOREM.

646. *The lateral or total areas of two similar cylinders of revolution* (§ 550) *are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*





Given S and s the lateral areas, T and t the total areas, V and v the volumes, H and h the altitudes, and R and r the radii of the bases, of two similar cylinders of revolution.

To Prove $\frac{S}{s} = \frac{T}{t} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

Proof. Since the generating rectangles are similar,

$$\frac{H}{h} = \frac{R}{r} \qquad (\S\ 253,\ 2)$$

$$= \frac{H + R}{h + r}. \qquad (\S\ 240)$$

$$\therefore \frac{S}{s} = \frac{2\pi RH}{2\pi rh} \ (\S\ 643) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2},$$

$$\frac{T}{t} = \frac{2\pi R(H + R)}{2\pi r(h + r)} \ (\S\ 643) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2},$$

and $$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} \ (\S\ 645) = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{H^3}{h^3}.$$

EXERCISES.

1. Find the lateral area, total area, and volume of a cylinder of revolution, the diameter of whose base is 18, and whose altitude is 16.

2. The radii of the bases of two similar cylinders of revolution are 24 and 44, respectively. If the lateral area of the first cylinder is 720, what is the lateral area of the second?

3. Find the altitude and diameter of the base of a cylinder of revolution, whose lateral area is 168π and volume 504π.

(Substitute the given values in the formulæ of §§ 643 and 645, and solve the resulting equations.)

4. Find the volume of a cylinder of revolution, whose total area is 170π and altitude 12.

5. How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in., and thickness 1 in.?

(Find the difference of the volumes of two cylinders of revolution. $\pi = 3.1416$.)

6. The cross-section of a tunnel, $2\frac{1}{2}$ miles in length, is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle; how many cu. yd. of material were taken out in its construction? ($\pi = 3.1416$.)

7. The volume of a cylinder of revolution is equal to its lateral area multiplied by one-half the radius of its base.

THE CONE.

DEFINITIONS.

647. The *lateral area* of a cone, or frustum of a cone, is the area of its lateral surface.

The *slant height* of a cone of revolution is the straight line drawn from the vertex to any point in the circumference of the base.

The *slant height* of a frustum of a cone of revolution is that portion of the slant height of the cone included between the bases of the frustum.

648. A pyramid is said to be *inscribed in a cone* when its lateral edges are elements of the conical surface; the base of the pyramid is inscribed in the base of the cone, and its vertex coincides with the vertex of the cone.

A pyramid is said to be *circumscribed about a cone* when its lateral faces are tangent to the cone, and its base lies in the same plane with the base of the cone; the base of the pyramid is circumscribed about the base of the cone, and its vertex coincides with the vertex of the cone.

649. A frustum of a pyramid is said to be *inscribed in a frustum of a cone* when its lateral edges are elements of the lateral surface of the frustum of the cone.

In this case, the bases of the frustum of the pyramid are inscribed in the bases of the frustum of the cone.

A frustum of a pyramid is said to be *circumscribed about a frustum of a cone* when its lateral faces are tangent to the frustum of the cone, and its bases lie in the same planes with the bases of the frustum of the cone.

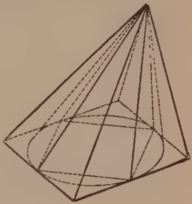
In this case, the bases of the frustum of the pyramid are circumscribed about the bases of the frustum of the cone.

650. It follows from § 363 that

If a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a circular cone (§ 553), *and the number of its faces be indefinitely increased,*

1. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*

2. *The volume of the pyramid approaches the volume of the cone as a limit.**



651. It follows from the above that

If a frustum of a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a frustum of a circular cone, and the number of its faces be indefinitely increased,

1. *The lateral area of the frustum of the pyramid approaches the lateral area of the frustum of the cone as a limit.*

2. *The volume of the frustum of the pyramid approaches the volume of the frustum of the cone as a limit.*

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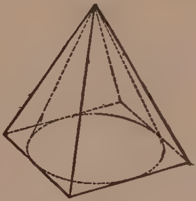
* For rigorous proofs of these statements, see Appendix, p. 388.

PROP. IV. THEOREM.

652. *The lateral area of a cone of revolution is equal to the circumference of its base, multiplied by one-half its slant height.*



Given S the lateral area, C the circumference of the base, and L the slant height, of a cone of revolution.

To Prove $S = C \times \frac{1}{2} L$.

Proof. Circumscribe about the cone a regular pyramid; let S' denote its lateral area, and C' the perimeter of its base.

Now the sides of the base of the pyramid are bisected at their points of contact with the base of the cone. (§ 174)

Then, the slant height of the pyramid is the same as the slant height of the cone. (§ 508)

$\therefore S' = C' \times \frac{1}{2} L$. (§ 512)

Now let the number of faces of the pyramid be indefinitely increased.

Then, S' approaches the limit S. (§ 650, 1)

And $C' \times \frac{1}{2} L$ approaches the limit $C \times \frac{1}{2} L$. (§ 363, I)

$\therefore S = C \times \frac{1}{2} L$. (?)

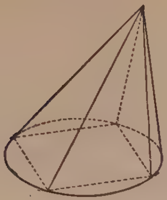
653. **Cor.** If S denotes the lateral area, T the total area, L the slant height, and R the radius of the base, of a cone of revolution,

$$S = 2\pi R \times \frac{1}{2} L \text{ (?) } = \pi RL.$$

And, $T = \pi RL + \pi R^2 \text{ (?) } = \pi R(L + R)$.

PROP. V. THEOREM.

654. *The volume of a circular cone is equal to the area of its base, multiplied by one-third its altitude.*



Given V the volume, B the area of the base, and H the altitude, of a circular cone.

To Prove $V = B \times \frac{1}{3} H.$

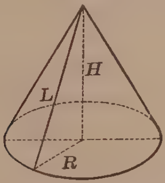
(Inscribe a pyramid whose base is a regular polygon.)

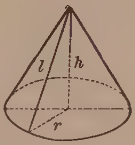
655. Cor. If V denotes the volume, H the altitude, and R the radius of the base, of a circular cone,

$$V = \frac{1}{3} \pi R^2 H. \qquad (?)$$

PROP. VI. THEOREM.

656. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as cubes of the radii of their bases.*





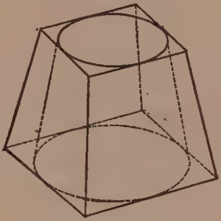
Given S and s the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution (§ 555).

To Prove $\frac{S}{s} = \frac{T}{t} = \frac{L^2}{l^2} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{L^3}{l^3} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

(The proof is left to the pupil; compare § 646.)

PROP. VII. THEOREM.

657. *The lateral area of a frustum of a cone of revolution is equal to the sum of the circumferences of its bases, multiplied by one-half its slant height.*



Given S the lateral area, C and c the circumferences of the bases, and L the slant height, of a frustum of a cone of revolution.

To Prove $S = (C + c) \times \frac{1}{2} L.$

Proof. Circumscribe about the frustum of the cone a frustum of a regular pyramid; let S' denote its lateral area, and C' and c' the perimeters of its bases.

Now the sides of the bases of the frustum of the pyramid are bisected at their points of contact with the bases of the frustum of the cone. (§ 174)

Then, the slant height of the frustum of the pyramid is the same as the slant height of the frustum of the cone. (§ 508)

$$\therefore\ S' = (C' + c') \times \tfrac{1}{2} L. \qquad (§ 513)$$

Now let the number of faces of the frustum of the pyramid be indefinitely increased.

Then, S' approaches the limit S, (§ 651, 1)
and $(C' + c') \times \frac{1}{2} L$ approaches the limit $(C + c) \times \frac{1}{2} L$. (§ 363, I)

$$\therefore\ S = (C + c) \times \tfrac{1}{2} L. \qquad (?)$$

658. Cor. I. If S denotes the lateral area, L the slant height, and R and r the radii of the bases, of a frustum of a cone of revolution,

$$S = (2\pi R + 2\pi r) \times \tfrac{1}{2} L \text{ (?)} = \pi(R + r)L.$$

659. Cor. II. We may write the first result of § 658

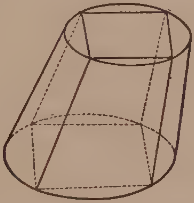
$$S = 2\pi \times \tfrac{1}{2}(R + r) \times L.$$

But, $2\pi \times \frac{1}{2}(R + r)$ is the circumference of a section equally distant from the bases. (§ 132)

Whence, *the lateral area of a frustum of a cone of revolution is equal to the circumference of a section equally distant from its bases, multiplied by its slant height.*

Prop. VIII. Theorem.

660. *The volume of a frustum of a circular cone is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Given V the volume, B and b the areas of the bases, and H the altitude, of a frustum of a circular cone.

To Prove $V = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H.$

(Inscribe a frustum of a pyramid whose base is a regular polygon. Then apply § 524.)

661. Cor. If V denotes the volume, H the altitude, and R and r the radii of the bases, of a frustum of a circular cone,

$$B = \pi R^2,\ b = \pi r^2, \text{ and } \sqrt{B \times b} = \sqrt{\pi^2 R^2 r^2} = \pi R r. \quad \text{(?)}$$

Then,

$$V = (\pi R^2 + \pi r^2 + \pi R r) \times \tfrac{1}{3} H = \tfrac{1}{3}\pi (R^2 + r^2 + Rr) H.$$

EXERCISES.

8. Find the lateral area, total area, and volume of a cone of revolution, the radius of whose base is 7, and whose slant height is 25.

9. Find the lateral area, total area, and volume of a frustum of a cone of revolution, the diameters of whose bases are 16 and 6, and whose altitude is 12.

10. The slant heights of two similar cones of revolution are 9 and 15, respectively. If the volume of the second cone is 625, what is the volume of the first?

11. Find the volume of a cone of revolution, whose slant height is 29 and lateral area 580π.

12. Find the lateral area of a cone of revolution, whose volume is 320π and altitude 15.

13. The altitude of a cone of revolution is 27, and the radius of its base is 16. What is the diameter of the base of an equivalent cylinder of revolution, whose altitude is 16?

14. The area of the entire surface of a frustum of a cone of revolution is 306π, and the radii of its bases are 11 and 5. Find its lateral area and volume.

15. The volume of a frustum of a cone of revolution is 6020π, its altitude is 60, and the radius of its lower base is 15. Find the radius of its upper base and its lateral area.

16. Find the altitude and lateral area of a cone of revolution, whose volume is 800π, and whose slant height is to the diameter of its base as 13 to 10.

17. The total areas of two similar cylinders of revolution are 32 and 162, respectively. If the volume of the second cylinder is 1458, what is the volume of the first?

(Let x and y denote the altitudes of the cylinders.)

18. The volumes of two similar cones of revolution are 343 and 512, respectively. If the lateral area of the first cone is 196, what is the lateral area of the second?

19. A cubical piece of lead, the area of whose entire surface is 384 sq. in., is melted and formed into a cone of revolution, the radius of whose base is 12 in. Find the altitude of the cone.

20. A tapering hollow iron column, 1 in. thick, is 24 ft. long, 10 in. in outside diameter at one end, and 8 in. in diameter at the other; how many cubic inches of metal were used in its construction?

(Find the difference of the volumes of the frustums of two cones of revolution. $\pi = 3.1416$.)

21. If the altitude of a cone of revolution is three-fourths the radius of its base, its volume is equal to its lateral area multiplied by one-fifth the radius of its base.

THE SPHERE.

DEFINITIONS.

662. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the circles which bound the zone are called the *bases*, and the perpendicular distance between their planes the *altitude*.

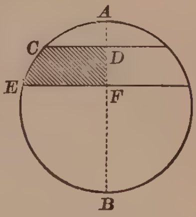
A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

A *spherical segment* is a portion of a sphere included between two parallel planes.

The circles which bound it are called the *bases*, and the perpendicular distance between them the *altitude*.

A *spherical segment of one base* is a spherical segment one of whose bounding planes is tangent to the sphere.

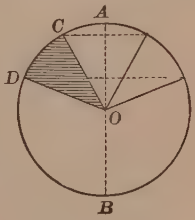
663. If semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are lines $\perp AB$, arc CE generates a zone whose altitude is DF, figure $CEFD$ a spherical segment whose altitude is DF, arc AC a zone of one base, and figure ACD a spherical segment of one base.



664. If a semicircle be revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a *spherical sector*.

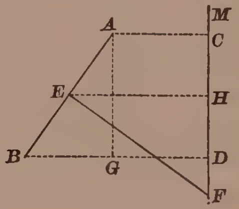
Thus, if semicircle $ACDB$ be revolved about diameter AB as an axis, sector OCD generates a spherical sector.

The zone generated by the arc of the sector is called the *base* of the spherical sector.



PROP. IX. THEOREM.

665. *The area of the surface generated by the revolution of a straight line about a straight line in its plane, not parallel to and not intersecting it, as an axis, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the middle point of the line and terminating in the axis.*



Given str. line AB revolved about str. line FM in its plane, not ∥ to and not intersecting it, as an axis; lines AC and $BD \perp FM$, and EF the ⊥ erected at the middle point of AB terminating in FM.

To Prove area $AB^* = CD \times 2\pi EF$. (§§ 276, 368)

Proof. Draw line $AG \perp BD$, and line $EH \perp CD$.

The surface generated by AB is the lateral surface of a frustum of a cone of revolution, whose bases are generated by AC and BD.

$$\therefore \text{area } AB = AB \times 2\pi EH. \qquad (\S\ 659)$$

But △ ABG and EFH are similar. (§ 262)

$$\therefore \frac{AB}{AG} = \frac{EF}{EH}. \qquad (?)$$

$$\therefore AB \times EH = AG \times EF \qquad (\S\ 232)$$

$$= CD \times EF. \qquad (?)$$

Substituting, we have

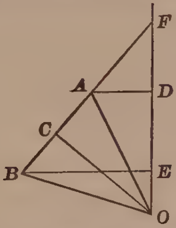
$$\text{area } AB = CD \times 2\pi EF.$$

---

* The expression "area AB" is used to denote *the area of the surface generated by AB.*

PROP. X. THEOREM.

666. *If an isosceles triangle be revolved about a straight line in its plane, not parallel to its base, as an axis, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.*



Given isosceles $\triangle\ OAB$ revolved about str. line OF in its plane, not ∥ to base AB, as an axis; and line $OC \perp AB$.

To Prove vol. OAB* = area $AB \times \frac{1}{3}\ OC$.

Proof. Draw lines AD and $BE \perp OF$; and produce BA to meet OF at F.

Now, vol. OBF = vol. OBE + vol. BEF

$$= \tfrac{1}{3}\pi\overline{BE}^2 \times OE + \tfrac{1}{3}\pi\overline{BE}^2 \times EF \qquad (\S\ 655)$$

$$= \tfrac{1}{3}\pi\overline{BE}^2 \times (OE + EF) = \tfrac{1}{3}\pi BE \times BE \times OF.$$

But $BE \times OF = OC \times BF$, for each expresses twice the area of $\triangle OBF$. (?)

$$\therefore \text{ vol. } OBF = \tfrac{1}{3}\pi BE \times OC \times BF.$$

But $\pi BE \times BF$ is the area of the surface generated by BF. (§ 653)

$$\therefore \text{ vol. } OBF = \text{area } BF \times \tfrac{1}{3}\ OC. \qquad (1)$$

Similarly, $\quad$ vol. OAF = area $AF \times \frac{1}{3}\ OC$. (2)

Subtracting (2) from (1), we have

$$\text{vol. } OAB = (\text{area } BF - \text{area } AF) \times \tfrac{1}{3}\ OC$$

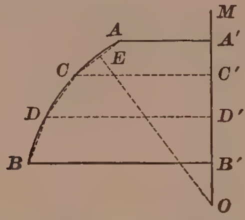
$$= \text{area } AB \times \tfrac{1}{3}\ OC.$$

---

* The expression "vol. OAB" is used to denote *the volume of the solid generated by OAB.*

PROP. XI. THEOREM.

667. *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*



Given arc AB revolved about diameter OM as an axis, lines AA' and $BB' \perp OM$, and R the radius of the arc.

To Prove area of zone generated by $AB = A'B' \times 2\pi R$.

Proof. Divide arc AB into three equal arcs, AC, CD, and DB, and draw chords AC, CD, and DB.

Also, draw lines CC' and $DD' \perp OM$, and line $OE \perp AC$.

$$\therefore \text{ area } AC = A'C' \times 2\pi OE,$$

$$\text{area } CD = C'D' \times 2\pi OE, \text{ etc.} \qquad (\S\ 665)$$

Adding these equations, we have

area of surface generated by broken line $ACDB$

$$= (A'C' + C'D' + \text{etc.}) \times 2\pi OE = A'B' \times 2\pi OE.$$

Now let the subdivisions of arc AB be bisected indefinitely.

Then, area of surface generated by broken line $ACDB$ approaches area of surface generated by arc AB as a limit. (§ 363, I*)

And, $A'B' \times 2\pi OE$ approaches $A'B' \times 2\pi R$ as a limit. (§ 364, 1*)

---

* The broken line $ACDB$ is called a *regular broken line*, and is said to be *inscribed in arc AB*; the theorems of §§ 363, I, and 364, 1, are evidently true when, instead of the perimeter of a regular inscribed polygon, we have a regular broken line inscribed in an arc.

For a rigorous proof of the statement that the area of the surface generated by $ACDB$ approaches the area of the surface generated by arc AB as a limit, see Appendix, p. 390.

Then, area of zone generated by arc $AB = A'B' \times 2\,\pi R$. (§ 188)

668. Sch. The proof of § 667 holds for any zone which lies entirely on the surface of a hemisphere; for, in that case, no chord is ∥ OM, and § 665 is applicable.

Since a zone which does not lie entirely on the surface of a hemisphere may be considered as the sum of two zones, each of which does lie entirely on the surface of a hemisphere, the theorem of § 667 is true for any zone.

669. Cor. I. If S denotes the area of a zone, h its altitude, and R the radius of the sphere,

$$S = 2\,\pi Rh.$$

670. Cor. II. Since the surface of a sphere may be regarded as a zone whose altitude is a diameter of the sphere, it follows that

The area of the surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.

671. Cor. III. Let S denote the area of the surface of a sphere, R its radius, and D its diameter.

Then, $S = 2\,R \times 2\,\pi R\ (?) = 4\,\pi R^2.$

That is, *the area of the surface of a sphere is equal to the square of its radius multiplied by* $4\,\pi$.

Again, $S = \pi \times (2\,R)^2 = \pi D^2.$

That is, *the area of the surface of a sphere is equal to the square of its diameter multiplied by* π.

672. Cor. IV. *The surface of a sphere is equivalent to four great circles.*

For πR^2 is the area of a great ⊙. (?)

673. Cor. V. *The areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

(The proof is left to the pupil; compare § 372.)

EXERCISES.

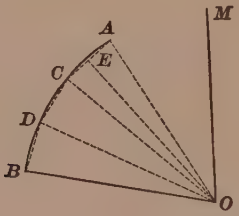
22. Find the area of the surface of a sphere whose radius is 12.

23. Find the area of a zone whose altitude is 13, if the radius of the sphere is 16.

24. Find the area of a spherical triangle whose angles are 125°, 133°, and 156°, on a sphere whose radius is 10.

PROP. XII. THEOREM.

674. *The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one-third the radius of the sphere.*



Given sector OAB revolved about diameter OM as an axis, and R the radius of the arc.

To Prove volume of spherical sector generated by OAB $=$ area of zone generated by $AB \times \frac{1}{3} R$.

Proof. Divide arc AB into three equal arcs, AC, CD, and DB, and draw chords AC, CD, and DB.

Also, draw lines OC and OD, and line $OE \perp AC$.

$$\therefore \text{ vol. } OAC = \text{area } AC \times \tfrac{1}{3} OE,$$

$$\text{vol. } OCD = \text{area } CD \times \tfrac{1}{3} OE, \text{ etc.} \qquad (\S\ 666)$$

Adding these equations, we have

$$\text{volume of solid generated by polygon } OACDB$$
$$= (\text{area } AC + \text{area } CD + \text{etc.}) \times \tfrac{1}{3} OE$$
$$= \text{area } ACDB \times \tfrac{1}{3} OE.$$

Now let the subdivisions of arc AB be bisected indefinitely.

Then, volume of solid generated by polygon $OACDB$ approaches volume of solid generated by sector OAB as a limit. (§ 363, II *)

And area of surface generated by $ACDB \times \frac{1}{3} OE$ approaches area of surface generated by arc $AB \times \frac{1}{3} R$ as a limit. (§§ 363, I, 364, 1 †)

Then, volume of solid generated by sector OAB

= area of zone generated by arc $AB \times \frac{1}{3} R$. (?)

675. Sch. It is evident, as in § 668, that the theorem of § 674 holds for any spherical sector.

676. Cor. I. If V denotes the volume of a spherical sector, h the altitude of the zone which forms its base, and R the radius of the sphere,

$$V = 2\pi Rh \times \tfrac{1}{3} R \,(\S\ 669) = \tfrac{2}{3}\pi R^2 h.$$

677. Cor. II. Since a sphere may be regarded as a spherical sector whose base is the surface of the sphere,

The volume of a sphere is equal to the area of its surface multiplied by one-third its radius.

678. Cor. III. Let V denote the volume of a sphere, R its radius, and D its diameter.

Then, $V = 4\pi R^2 \times \frac{1}{3} R \,(\S\ 671) = \frac{4}{3}\pi R^3$.

That is, *the volume of a sphere is equal to the cube of its radius multiplied by* $\frac{4}{3}\pi$.

Again, $V = \pi D^2 \times \frac{1}{6} D \,(\S\ 671) = \frac{1}{6}\pi D^3$.

That is, *the volume of a sphere is equal to the cube of its diameter multiplied by* $\frac{1}{6}\pi$.

---

* The polygon $OACDB$ is called a *regular polygonal sector*, and is said to be *inscribed in sector* OAB; the theorem of § 363, II, is evidently true when, instead of a regular inscribed polygon, we have a regular polygonal sector inscribed in a sector.

For a rigorous proof of the statement that the volume of the solid generated by $OACDB$ approaches the volume of the solid generated by sector OAB as a limit, see Appendix, p. 391.

† See note foot of p. 374.

679. Cor. IV. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

(The proof is left to the pupil.)

680. Cor. V. *The volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

Given P the volume of a spherical pyramid, K the area of its base, and R the radius of the sphere.

To Prove $P = K \times \frac{1}{3} R.$

Proof. Let n denote the number of sides of the base of the spherical pyramid, s the sum of its $\angle$s referred to a rt. $\angle$ as the unit of measure, T the area of a tri-rectangular △, T'' the volume of a tri-rectangular pyramid, S the area of the surface of the sphere, and V its volume.

Then, $$\frac{P}{K} = \frac{[s-2(n-2)] \times T'}{[s-2(n-2)] \times T} = \frac{T''}{T}. \quad (\S\S\ 636, 637)$$

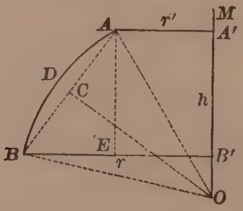
Also, $$\frac{V}{S} = \frac{8\,T''}{8\,T} = \frac{T''}{T}. \quad (\S\ 609)$$

$$\therefore\ \frac{P}{K} = \frac{V}{S} = \frac{\frac{4}{3}\pi R^3}{4\pi R^2}\ (\S\S\ 671, 678) = \tfrac{1}{3} R.$$

$$\therefore\ P = K \times \tfrac{1}{3} R.$$

Prop. XIII. Problem.

681. *Given the radii of the bases, and the altitude, of a spherical segment, to find its volume.*



Given O the centre of arc ADB, lines AA' and $BB' \perp$ to diameter OM, $AA' = r'$, $BB' = r$, $A'B' = h$, and figure $ADBB'A'$ revolved about OM as an axis.

Required to express volume of spherical segment generated by $ADBB'A'$ in terms of r, r', and h.

Solution. Draw lines OA, OB, and AB; also, line $OC \perp AB$, and line $AE \perp BB'$; and denote radius OA by R.

Now, vol. $ADBB'A' =$ vol. $ACBD +$ vol. $ABB'A'$. (1)

Also, vol. $ACBD =$ vol. $OADB -$ vol. OAB.

But, vol. $OADB = \frac{2}{3}\pi R^2 h$. (§ 676)

And, vol. $OAB =$ area $AB \times \frac{1}{3} OC$ (§ 666)

$= h \times 2\pi OC \times \frac{1}{3} OC$ (§ 665)

$= \frac{2}{3}\pi \overline{OC}^2 h$.

$\therefore$ vol. $ACDB = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi \overline{OC}^2 h$

$= \frac{2}{3}\pi (R^2 - \overline{OC}^2) h$.

But, $R^2 - \overline{OC}^2 = \overline{AC}^2$ (§ 273)

$= (\frac{1}{2} AB)^2$ (?)

$= \frac{1}{4}\overline{AB}^2$.

$\therefore$ vol. $ACDB = \frac{2}{3}\pi \times \frac{1}{4}\overline{AB}^2 \times h = \frac{1}{6}\pi \overline{AB}^2 h$.

Now, $\overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2$ (?)

$= (r - r')^2 + h^2$. (?)

$\therefore$ vol. $ACDB = \frac{1}{6}\pi [(r - r')^2 + h^2] h$.

Also, vol. $ABB'A' = \frac{1}{3}\pi (r^2 + r'^2 + rr') h$. (§ 661)

Substituting in (1), we have

vol. $ADBB'A'$

$$= \frac{1}{6}\pi [(r - r')^2 + h^2] h + \frac{1}{6}\pi (2r^2 + 2r'^2 + 2rr') h$$

$$= \frac{1}{6}\pi (r^2 - 2rr' + r'^2 + h^2 + 2r^2 + 2r'^2 + 2rr') h$$

$$= \frac{1}{6}\pi (3r^2 + 3r'^2) h + \frac{1}{6}\pi h^3$$

$$= \frac{1}{2}\pi (r^2 + r'^2) h + \frac{1}{6}\pi h^3.$$

682. Cor. If r denotes the radius of the base, and h the altitude, of a spherical segment of one base, its volume is

$$\tfrac{1}{2}\pi r^2 h + \tfrac{1}{6}\pi h^3.$$

EXERCISES.

25. Find the volume of a sphere whose radius is 12.

26. Find the volume of a spherical sector, the altitude of whose base is 12, the diameter of the sphere being 25.

27. Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

28. Find the radius and volume of a sphere, the area of whose surface is 324π.

29. Find the diameter and area of the surface of a sphere whose volume is $\frac{1125}{2}\pi$.

30. The surface of a sphere is equivalent to the lateral surface of its circumscribed cylinder.

31. The volume of a sphere is two-thirds the volume of its circumscribed cylinder.

32. A spherical cannon-ball 9 in. in diameter is dropped into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box? ($\pi = 3.1416$.)

33. What is the angle of the base of a spherical wedge whose volume is $\frac{40}{3}\pi$, if the radius of the sphere is 4?

34. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°; the diameter of the sphere being 12.

35. The surface of a sphere is equivalent to two-thirds the entire surface of its circumscribed cylinder.

36. Prove Prop. IX. when the straight line is parallel to the axis.

37. Find the area of the surface and the volume of a sphere inscribed in a cube the area of whose surface is 486.

38. How many spherical bullets, each $\frac{5}{8}$ in. in diameter, can be formed from five pieces of lead, each in the form of a cone of revolution, the radius of whose base is 5 in., and whose altitude is 8 in.?

39. A cylindrical vessel, 8 in. in diameter, is filled to the brim with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{4}$ in. Find the diameter of the ball.

40. If a sphere 6 in. in diameter weighs 351 ounces, what is the weight of a sphere of the same material whose diameter is 10 in.?

41. If a sphere whose radius is $12\frac{1}{2}$ in. weighs 3125 lb., what is the radius of a sphere of the same material whose weight is $819\frac{1}{5}$ lb.?

42. The altitude of a frustum of a cone of revolution is $3\frac{1}{2}$, and the radii of its bases are 5 and 3; what is the diameter of an equivalent sphere?

43. Find the radius of a sphere whose surface is equivalent to the entire surface of a cylinder of revolution, whose altitude is $10\frac{1}{2}$, and radius of base 3.

44. The volume of a cylinder of revolution is equal to the area of its generating rectangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the centre of the rectangle.

45. The volume of a cone of revolution is equal to its lateral area, multiplied by one-third the perpendicular from the vertex of the right angle to the hypotenuse of the generating triangle.

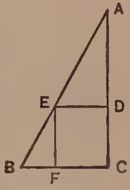
46. Two zones on the same sphere, or equal spheres, are to each other as their altitudes.

47. The area of a zone of one base is equal to the area of the circle whose radius is the chord of its generating arc. (§ 270, 2.)

48. If the radius of a sphere is R, what is the area of a zone of one base, whose generating arc is 45°? (Ex. 55, p. 210.)

49. If the altitude of a cone of revolution is 15, and its slant height 17, find the total area of an inscribed cylinder, the radius of whose base is 5.

(Let the cone and cylinder be generated by the revolution of rt. $\triangle ABC$ and rect. $CDEF$ about AC as an axis.)



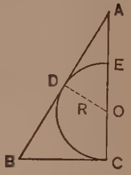
50. Find the area of the surface and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 9, and whose altitude is 24.

51. An equilateral triangle, whose side is 6, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume, of the solid generated.

52. A cone of revolution is inscribed in a sphere whose diameter is $\frac{4}{3}$ the altitude of the cone. Prove that its lateral surface and volume are, respectively, $\frac{3}{8}$ and $\frac{9}{32}$ the surface and volume of the sphere.

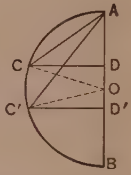
53. Find the volume of a sphere circumscribing a cube whose volume is 64.

54. A cone of revolution is circumscribed about a sphere whose diameter is two-thirds the altitude of the cone. Prove that its lateral surface and volume are, respectively, three-halves and nine-fourths the surface and volume of the sphere.



55. If the radius of a sphere is 25, find the lateral area and volume of an inscribed cone, the radius of whose base is 24.

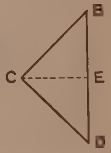
(Two solutions.)



56. If the volume of a sphere is $\frac{500}{3}\pi$, find the lateral area and volume of a circumscribed cone whose altitude is 18.

57. Find the volume of a spherical segment of one base whose altitude is 6, the diameter of the sphere being 30.

58. A square whose area is A revolves about its diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.



59. The altitude of a cone of revolution is 9. At what distances from the vertex must it be cut by planes parallel to its base, in order that it may be divided into three equivalent parts? (§ 656.)

(Let V denote the volume of the cone, x the distance from the vertex to the nearer plane, and y the distance to the other.)

60. Given the radius of the base, R, and the total area, T, of a cylinder of revolution, to find its volume.

(Find H from the equation $T = 2\pi RH + 2\pi R^2$.)

61. Given the diameter of the base, D, and the volume, V, of a cylinder of revolution, to find its lateral area and total area.

62. Given the altitude, H, and the volume, V, of a cone of revolution, to find its lateral area.

63. Given the slant height, L, and the lateral area, S, of a cone of revolution, to find its volume.

64. A circular sector whose central angle is 45° and radius 12 revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

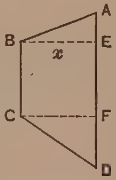
65. Given the area of the surface of a sphere, S, to find its volume.

66. Given the volume of a sphere, V, to find the area of its surface.

67. A right triangle, whose legs are a and b, revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume, of the solid generated.

68. The parallel sides of a trapezoid are 12 and 26, respectively, and its non-parallel sides are 13 and 15. Find the volume generated by the revolution of the trapezoid about its longest side as an axis.

(Represent BE by x.)



69. An equilateral triangle, whose altitude is h, revolves about one of its altitudes as an axis. Find the area of the surface, and the volume, of the solids generated by the triangle, and by its inscribed circle. (Ex. 21, p. 151.)

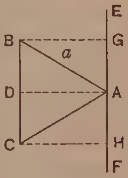
70. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a cone of revolution whose altitude is h, and radius of base r.

(Represent altitude of cylinder by x.)

71. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a sphere whose radius is r.

72. An equilateral triangle, whose side is a, revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume, of the solid generated.

(The solid generated is the difference of the cylinder generated by $BCHG$, and the cones generated by ABG and ACH.)



73. The outer diameter of a spherical shell is 9 in., and its thickness is 1 in. What is its weight, if a cubic inch of the metal weighs $\frac{1}{3}$ lb.? ($\pi = 3.1416$.)

74. Find the diameter of a sphere in which the area of the surface and the volume are expressed by the same numbers.

75. A regular hexagon, whose side is a, revolves about its longest diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.

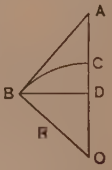
76. The sides AB and BC of rectangle $ABCD$ are 5 and 8, respectively. Find the volumes generated by the revolution of triangle ACD about sides AB and BC as axes.

77. The sides of a triangle are 17, 25, and 28. Find the volume generated by the revolution of the triangle about its longest side as an axis. (§ 324.)

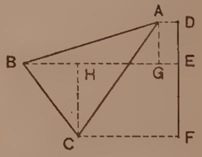
78. A frustum of a circular cone is equivalent to three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum. (§ 660.)

79. The volume of a cone of revolution is equal to the area of its generating triangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the intersection of the medians of the triangle. (§ 140.)

80. If the earth be regarded as a sphere whose radius is R, what is the area of the zone visible from a point whose height above the surface is H? (§ 271, 2.)

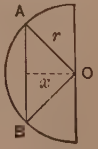


81. The sides AB and BC of acute-angled triangle ABC are $\sqrt{241}$ and 10, respectively. Find the volume of the solid generated by the revolution of the triangle about an axis in its plane, not intersecting its surface, whose distances from A, B, and C are 2, 17, and 11, respectively.



82. A projectile consists of two hemispheres, connected by a cylinder of revolution. If the altitude and diameter of the base of the cylinder are 8 in. and 7 in., respectively, find the number of cubic inches in the projectile. ($\pi = 3.1416$.)

83. A segment of a circle, whose bounding arc is a quadrant, and whose radius is r, revolves about a diameter parallel to its bounding chord. Find the area of the entire surface, and the volume, of the solid generated.



84. If any triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.

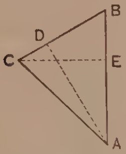


Fig. 1.

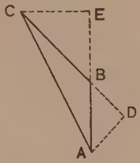


Fig. 2.

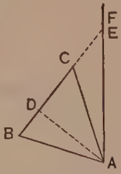


Fig. 3.

(Compare § 666. Case I., Figs. 1 and 2, when a side coincides with the axis; there are two cases according as AD falls on BC, or BC produced. Case II., Fig. 3, when no side coincides with the axis; prove by Case I.)

85. If any triangle be revolved about an axis which passes through its vertex parallel to its base, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.

(Compare Ex. 72, p. 383. There are two cases according as AD falls on BC, or BC produced.)

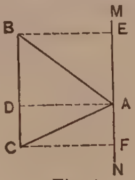


Fig. 1.

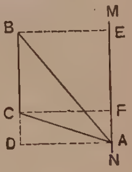
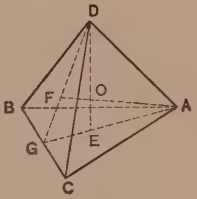


Fig. 2.

86. Find the area of the surface of the sphere circumscribing a regular tetraedron, whose edge is 8.

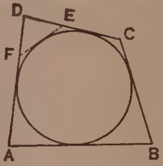
(Draw lines DOE and AOF $\perp$ to $\triangle s$ ABC and BCD, respectively.)



APPENDIX.

PROOF OF STATEMENT MADE IN ELEVENTH LINE, PAGE 201.

683. Theorem. *The circumference of a circle is shorter than the perimeter of any circumscribed polygon.*



Given polygon $ABCD$ circumscribed about a ⊙.

To Prove circumference of ⊙ shorter than perimeter $ABCD$.

Proof. Of the perimeters of the ⊙ and of its circumscribed polygons, there must be one perimeter such that all the others are of equal or greater length.

But no circumscribed polygon can have this perimeter.

For, if we suppose polygon $ABCD$ to have this perimeter, and draw a tangent to the ⊙, meeting CD and DA at points E and F, respectively, then since str. line EF is $<$ broken line EDF, the perimeter of circumscribed polygon $ABCEF$ is $<$ perimeter $ABCD$.

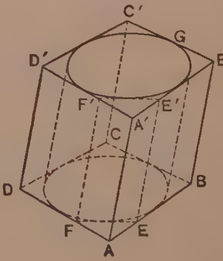
Hence, the circumference of the ⊙ is $<$ the perimeter of any circumscribed polygon.

PROOFS OF THE LIMIT STATEMENTS OF § 640.

684. We assume the following:

A portion of a plane is less than any other surface having the same boundaries.

685. Theorem. *The total surface of a circular cylinder is less than the total surface of any circumscribed prism.*



Given prism AC' circumscribed about circular cylinder EG.

To Prove total surface $EG <$ total surface AC'.

Proof. Of the total surfaces of the cylinder and of its circumscribed prisms, there must be one total surface such that the area of every other is either equal to or $>$ it.

But no circumscribed prism can have this total surface.

For suppose prism AC' to have this total surface; and let $BCDFE - E'$ be a circumscribed prism, whose face EF' intersects faces AB' and AD' in lines EE' and FF', respectively.

Now, face EF' is $<$ sum of faces AE', AF', AEF, and $A'E'F'$. (§ 684)

Whence, total surface of prism $BCDFE - E'$ is $<$ total surface of prism AC'.

Then, total surface of cylinder EG is $<$ total surface of any circumscribed prism.

PROOFS OF THE LIMIT STATEMENTS OF § 640.

686. Let L denote the lateral edge, H the altitude, S and s the the lateral areas, V and v the volumes, E and e the perimeters of rt. sections, and B and b the areas of the bases of the circumscribed and inscribed prisms, respectively; also, S' the lateral area of the cylinder, V' its volume, E' the perimeter of a rt. section, and B' the area of the base.

1. We have, $S + 2B > S' + 2B'$. (§ 685)

$$\therefore S + 2(B - B') > S'.$$

Again, the total surface of the inscribed prism is $<$ the total surface of the cylinder. (§ 684)

$$\therefore S' + 2B' > s + 2b, \text{ or } S' > s + 2(b - B').$$

Then, $S + 2(B - B') > S' > s + 2(b - B')$.

Now if the number of faces of the prisms be indefinitely increased, $B - B'$ and $b - B'$ approach the limit 0. (§ 363, II)

Again, the difference between the perimeters of the bases of the prisms approaches the limit 0. (§ 363, I)

Then, the total surface of the circumscribed prism continually decreases, but never reaches the total surface of the inscribed prism; and the total surface of the inscribed prism continually increases, but never reaches the total surface of the circumscribed prism. (§ 684)

Then, the difference between $S + 2B$ and $s + 2b$ can be made less than any assigned value, however small.

Whence, $S + 2B - (s + 2b)$, or $S - s + 2(B - b)$, approaches the limit 0.

But $B - b$ approaches the limit 0. (§ 363, II)

Whence, $S - s$ approaches the limit 0.

Then, S' is intermediate in value between two variables, the difference between which approaches the limit 0.

Then, the difference between either variable and S', that is,

$$S + 2(B - B') - S' \text{ and } S' - s - 2(b - B'),$$

approaches the limit 0.

Whence, $S - S'$ and $S' - s$ approach the limit 0.

Hence, S and s approach the limit S'.

2. We have, $V = B \times H$ and $v = b \times H$. (§ 499)

Whence, $V - v = B \times H - b \times H = (B - b) \times H$.

Now if the number of faces of the prisms be indefinitely increased, $B - b$, and therefore $V - v$, approaches the limit 0. (§ 363, II)

But V' is evidently $> v$, and $< V$.

Then, $V - V'$ and $V' - v$ approach the limit 0.

Whence, V and v approach the limit V'.

3. We have, $S = E \times L$ and $s = e \times L$. (§ 484)

Then, $E = \frac{S}{L}$ and $e = \frac{s}{L}$; or, $E - e = \frac{S - s}{L}$.

Now if the number of faces of the prisms be indefinitely increased, $S - s$, and therefore $E - e$, approaches the limit 0. (§ 640, 1)

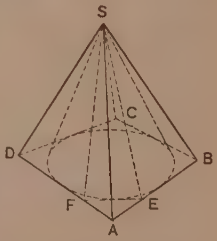
But E', the perimeter of a rt. section of the cylinder, is $< E$; for the theorem of § 683 is evidently true when for the ⊙ is taken any closed curve whose tangents do not intersect its surface; also, E' is $> e$. (Ax. 4)

Then, $E - E'$ and $E' - e$ approach the limit 0.

Whence, E and e approach the limit E'.

PROOFS OF THE LIMIT STATEMENTS OF § 650.

687. Theorem. *The total surface of a circular cone is less than the total surface of any circumscribed pyramid.*



Given pyramid $S\text{-}ABCD$ circumscribed about circular cone $S\text{-}EF$.

To Prove total surface $S\text{-}EF <$ total surface $S\text{-}ABCD$.

Proof. Of the total surfaces of the cone and of its circumscribed pyramids, there must be one total surface such that the area of every other is either equal to or $>$ it.

But no circumscribed pyramid can have this total surface.

For suppose pyramid $S\text{-}ABCD$ to have this total surface; and let $S\text{-}BCDFE$ be a circumscribed pyramid, whose face SEF intersects faces SAB and SAD in lines SE and SF, respectively.

Now, face SEF is $<$ sum of faces SAE, SAF, and AEF. (§ 684)

Whence, total surface of pyramid $S\text{-}BCDFE$ is $<$ total surface of pyramid $S\text{-}ABCD$.

Then, total surface of cone $S\text{-}EF$ is $<$ total surface of any circumscribed pyramid.

Proofs of the Limit Statements of § 650.

688. Let H denote the altitude, S and s the lateral areas, V and v the volumes, and B and b the areas of the bases, of the circumscribed and inscribed pyramids, respectively; also, S' the lateral area of the cone, V' its volume, and B' the area of its base.

1. We have, $$S + B > S' + B'. \qquad (§ 687)$$

$$\therefore\ S + (B - B') > S'.$$

Again, the total surface of the inscribed pyramid is $<$ the total surface of the cone. (§ 684)

$$\therefore\ S' + B' > s + b, \text{ or } S' > s + (b - B').$$

Then, $$S + (B - B') > S' > s + (b - B').$$

Now if the number of faces of the pyramids be indefinitely increased, $B - B'$ and $b - B'$ approach the limit 0. (§ 363, II)

Also, the difference between the perimeters of the bases of the pyramids approaches the limit 0. (§ 363, I)

Then, $S + B$ continually decreases, and $s + b$ continually increases; and the difference between them can be made less than any assigned value, however small. (§ 684)

Then, $S - s + (B - b)$ approaches the limit 0.

But $B - b$ approaches the limit 0. (§ 363, II)

Whence, $S - s$ approaches the limit 0.

Then, S' is intermediate in value between two variables, the difference between which approaches the limit 0.

Whence, the difference between either variable and S', that is, $S + (B - B') - S'$ and $S' - s - (b - B')$, approaches the limit 0.

Then, $S - S'$ and $S' - s$ approach the limit 0.

Whence, S and s approach the limit S'.

2. We have, $$V = B \times \tfrac{1}{3} H \text{ and } v = b \times \tfrac{1}{3} H. \qquad (§ 521)$$

Whence, $$V - v = (B - b) \times \tfrac{1}{3} H.$$

Now if the number of faces of the pyramids be indefinitely increased, $B - b$, and therefore $V - v$, approaches the limit 0. (§ 363, II)

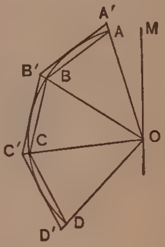
But, V' is evidently $> v$, and $< V$.

Then, $V - V$ and $V' - v$ approach the limit 0.

Whence, V and v approach the limit V'.

PROOF OF THE LIMIT STATEMENT IN NOTE FOOT OF PAGE 374.

689. Theorem. *If a regular broken line, inscribed in an arc, be revolved about a diameter, not intersecting the arc, as an axis, and the subdivisions of the arc be bisected indefinitely, the area of the surface generated by the broken line approaches the area of the surface generated by the arc as a limit.*



Given regular broken line $ABCD$, inscribed in arc AD, revolving about diameter OM as an axis.

To Prove that, if the subdivisions of arc AD be bisected indefinitely, area of surface generated by $ABCD$ approaches area of surface generated by arc AD as a limit.

Proof. Let $A'B'$, $B'C'$, and $C'D'$ be tangents ∥ to AB, BC, and CD, respectively, points A', B', C', and D' being in radii OA, OB, OC, and OD, respectively, produced; and let S, s, and S' denote the areas of the surfaces generated by $A'B'C'D'$, and $ABCD$, and arc AD, respectively.

Of the surfaces generated by arc AD, by $ABCD$, and by regular inscribed broken lines obtained by bisecting the subdivisions of the arc indefinitely, there must be one surface such that the areas of all the others are either equal to or $<$ it.

But no regular inscribed broken line can generate this surface.

For if this were the case, by bisecting the subdivisions of the arc, a regular inscribed broken line would be obtained having the same projection on the axis; but the ⊥ from O to each line would be greater, and hence the surface generated would be greater.

(§ 665, and Note foot of p. 374.)

Hence, surface generated by arc AD is $>$ surface generated by $ABCD$; that is, S' is $> s$.

Again, of the surfaces generated by arc AD, by $A'B'C'D'$, and by regular circumscribed broken lines obtained by bisecting the subdivisions of the arc indefinitely, there must be one surface such that the areas of all the others are either equal to or $>$ it.

But no regular circumscribed broken line can generate this surface.

For if this were the case, by bisecting the subdivisions of the arc, a regular circumscribed broken line would be obtained in which the ⊥ from O to each line would be the same; but the projection on the axis would be smaller, and hence the surface generated would be smaller.

Hence, surface generated by arc AD is $<$ surface generated by $A'B'C'D'$; that is, S' is $< S$.

Then, $S - S'$ and $S' - s$ are $< S - s$.

Now if the subdivisions of arc AD be bisected indefinitely, the difference between broken lines $A'B'C'D'$ and $ABCD$ approaches the limit 0. (Note foot p. 374.)

Then, the difference between the projections on OM of $A'B'C'D'$ and $ABCD$ approaches the limit 0.

Also, the difference between the ⊥s from O to $A'B'$ and AB approaches the limit 0. (Note foot p. 374.)

Then, the difference between the areas of the surfaces generated by $A'B'C'D'$ and $ABCD$, that is, $S - s$ approaches the limit 0. (§ 665)

Then, $S - S'$ and $S' - s$ approach the limit 0.

Whence, S and s approach the limit S'.

PROOF OF THE LIMIT STATEMENT IN NOTE FOOT OF PAGE 377.

690. Theorem. *If a regular polygonal sector, inscribed in a sector of a circle, be revolved about a diameter, not crossing the sector, as an axis, and the subdivisions of the arc be bisected indefinitely, the volume of the solid generated by the polygonal sector approaches the volume of the solid generated by the sector as a limit.*

Given regular polygonal sector $OABCD$, inscribed in sector OAD, revolved about diameter OM as an axis. (Fig. of § 689.)

To Prove that, if the subdivisions of arc AD be bisected indefinitely, volume of solid generated by $OABCD$ approaches volume of solid generated by sector OAD as a limit.

Proof. Let $A'B'$, $B'C'$, and $C'D'$ be tangents ∥ to AB, BC, and CD, respectively, points A', B', C', and D' being in radii OA, OB, OC, and OD, respectively, produced; and let V, v, and V' denote the volumes of the solids generated by $OA'B'C'D'$, $OABCD$, and sector OAD, respectively.

Then, V' is evidently $> v$, and $< V$.

Whence, $V - V'$ and $V' - v$ are $< V - v$.

Now if the subdivisions of arc AD be bisected indefinitely, the difference between the areas of $OA'B'C'D'$ and $OABCD$, and therefore $V - v$, approaches the limit 0. (Note foot p. 377.)

Then, $V - V'$ and $V' - v$ approach the limit 0.

Whence, V and v approach the limit V'.

SUGGESTIONS FOR THE USE OF COLORED PLATES.

In beginning the study of solid geometry, a new difficulty is encountered, the difficulty of seeing the figures correctly. Through plane geometry, the pupil has acquired the habit of looking at the figures simply as lines making different angles and running in varying directions, but always limited to one plane. To the untrained eye, the line figures in solid geometry do not look essentially different. The teacher sees, the pupil does not, and, worse than all, too frequently the teacher fails to realize that what represents to him a solid figure is, to the pupil, a number of lines similar to those in plane geometry, only hopelessly complicated in arrangement.

The first thing necessary, then, is to train a class to visualize correctly, — to see in imagination, not a seeming confusion of lines, but the solids outlined by those lines.

In the hope of accomplishing this, various aids have been offered by text-books in the way of graphic representation, but all of them, while attractive at first, have, when tried, fallen short of expectation in teaching value.

Work with actual models is accurate and helpful, but photographic reproductions of these models make nearly the same demands upon the untrained imagination that the line figures do. Shaded figures have been used, but the similarity in tone of grays and blacks is confusing to the uneducated eye.

With the color scheme here presented, the confusion vanishes, and the pupil not only *may* see, but *must* see, the planes in their true relations to each other. If his first glimpse of figures for solids is right, he is ready then to look for depth, distance, and three dimensions, in all succeeding figures.

The few colored figures here presented are valuable in the beginning, to show the pupil the kind of thing that he is to look for — what he is expected to see. Take, for instance, the figure on page 256. To the beginner there is little suggestion of various planes intersecting, disappearing behind each other, and reappearing, but by Plate I all this is instantly revealed. The correct visual impression here gained will then be transferred naturally to the line figure.

Another objection to the aids thus far presented lies in the fact that the text-book does all the work, leaving the pupil only an observer. If the work stops with looking at the figures and studying from them, their greatest teaching value is lost. It is comparatively easy, with a figure that has been carefully drawn and effectively shaded or colored, to grasp for the moment the general idea indicated, but the impression will be neither complete nor lasting.

Purposely only a few suggestive figures are here presented in color, it being the plan that the pupil, from the figures given in the text, and from the accompanying demonstration, shall interpret in color the solids indicated. When he is compelled thus to fix the limitations of the planes, he is led to definite knowledge that is otherwise impossible. Here is represented all the vast distance that lies between looking at a picture done by some one else, and reproducing that picture yourself, — all the difference between observing and doing.

The plan here offered is capable of practical application in several ways.

Send the class to the board to draw the figures for the day with colored crayons; the result will reveal their understanding or misunderstanding of the proposition under consideration. With the color in their own hands, pupils are compelled to decide where plane intersects plane, where one disappears behind another, and many other things that escaped their observation in studying from the book. Take, for instance, the figure on page 256. It probably never occurred to the pupil in studying to observe into how many planes the argument is carried. When he colors it he must know.

This first work should be done rapidly, with no attempt at finished drawings. Sometimes it is well to have a class draw entirely free-hand, laying in the color rapidly, attempting only to bring out the geometric idea. At first strongly contrasting colors should be used, and, as the work is not permanent, they may be even crude, if only striking.

Following this, certain figures should be put into permanent form. Such drawing should be done carefully, with as close mathematical accuracy as board and chalk will allow. Here attention should be given to the color scheme and the result made restful to the eye. There is actual teaching value in having these difficult figures long before the attention.

With the figures thus before the class, it is easy in the few spare moments that occasionally come at the close of a recitation, to give a quick review that would not be possible if time had to be consumed in drawing.

Blackboard work is strengthened by outlining all planes in strong white lines, just as in the book they are outlined in black.

Another useful expedient in the use of color is the careful preparation of plates outside of class. The most interesting and effective figures should be selected, and all members of the class required to execute a certain number, this number varying according to the ability of different classes. It is usually well to suggest a uniform size of paper; seven by nine inches, approximately, is desirable. One figure only should be placed on a sheet. As to size of figure, it is better not to dictate. When the first drawings are brought together, it will not take a class long to decide which is most effective. From this they will modify their scale, approximating the best, but retaining perfect individuality.

One additional direction should be given, applying equally to board work and to work on paper. Leave only such lines as would be visible if the planes were opaque. A glance at the colored plates here given will show that by omitting the dotted lines the figure is more effective, and the solidarity greatly emphasized. The geometric value of the lines is not lost, for the eye naturally carries them along behind the plane, and joins the parts correctly if they reappear. In outlining the figure at first, these lines, of course, should be drawn, for by them is frequently determined where the planes intersect.

As to the medium used, that is a matter of taste and equipment. Colored pencils are easiest for the untrained hand, and good effects can be obtained with them. If any one in the class handles water colors, he should be encouraged to use them, for they make stronger figures, and the influence of even one or two working in them will elevate the entire standard.

Some or all of these expedients may be used as the conditions of individual classes indicate, but let it always be insisted that the *class* do the work. The most carefully executed drawing of teacher or text-book is worth less than the poorest attempt of the poorest pupil.

Finally, the method here presented is offered only as a practical suggestion for clearer teaching, not as an integral part of geometry, and may be used or not as teachers desire. Like everything else, it is capable of abuse and perversion, and whoever uses it should be ever watchful lest it overstep its proper limitations. Its purpose is not to produce a fine set of drawings, but to assist in teaching geometry. It is a means, not an end; an expedient, not a science.

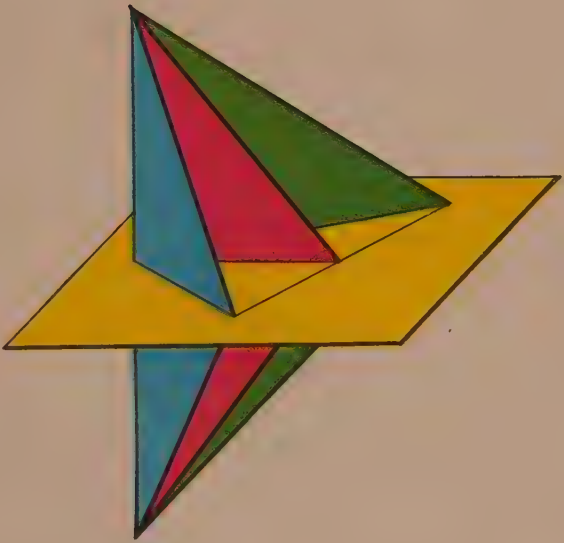


Plate I.

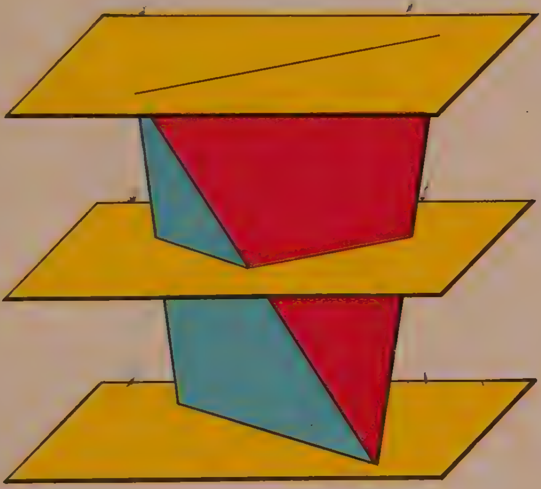


Plate II.

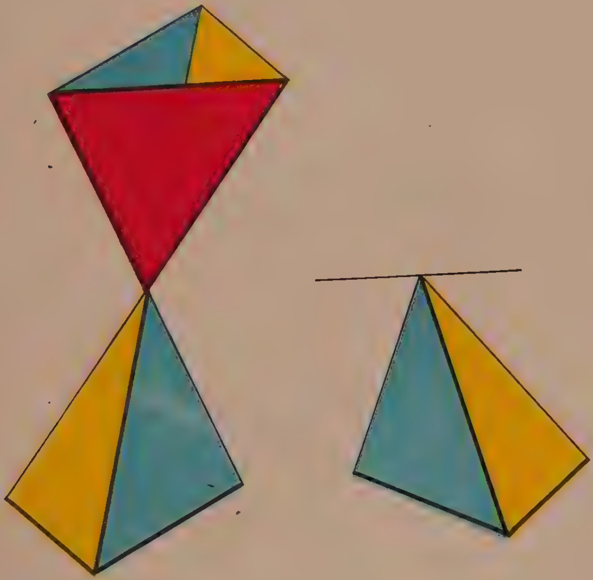


Plate III.

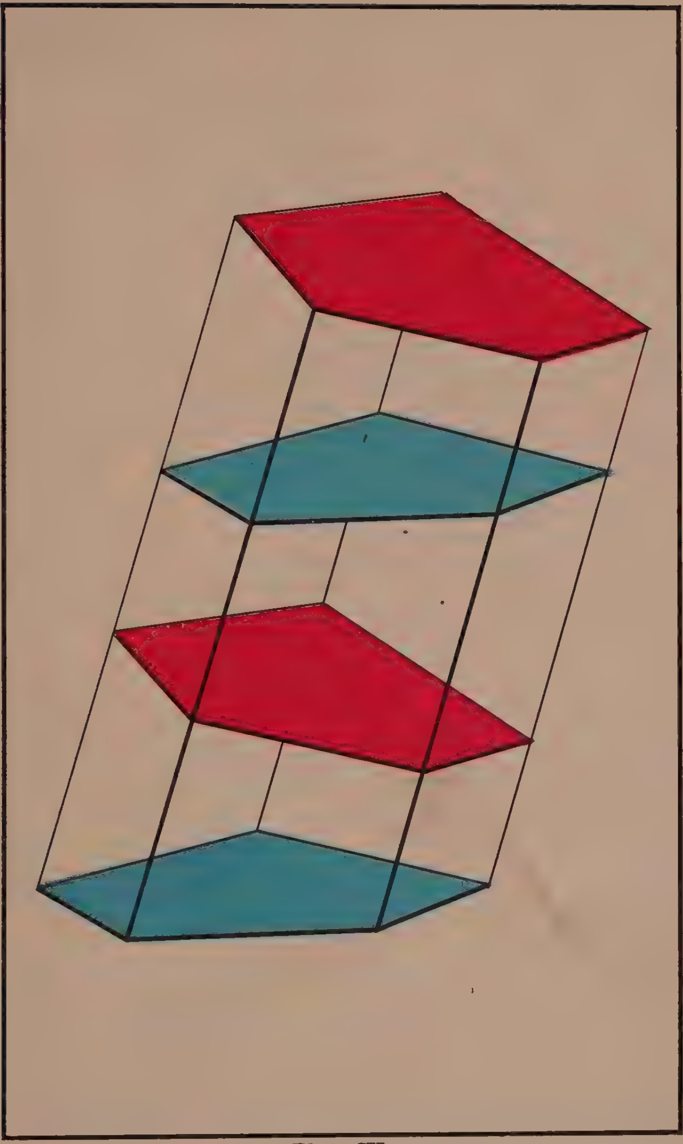


Plate IV.

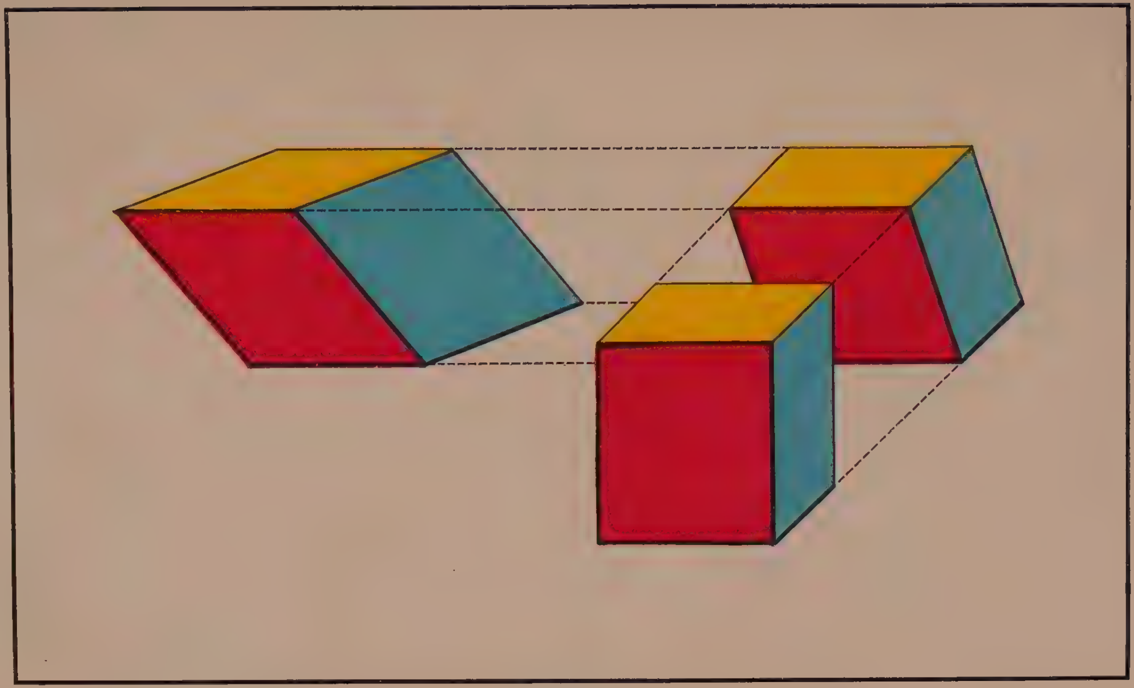


Plate V.

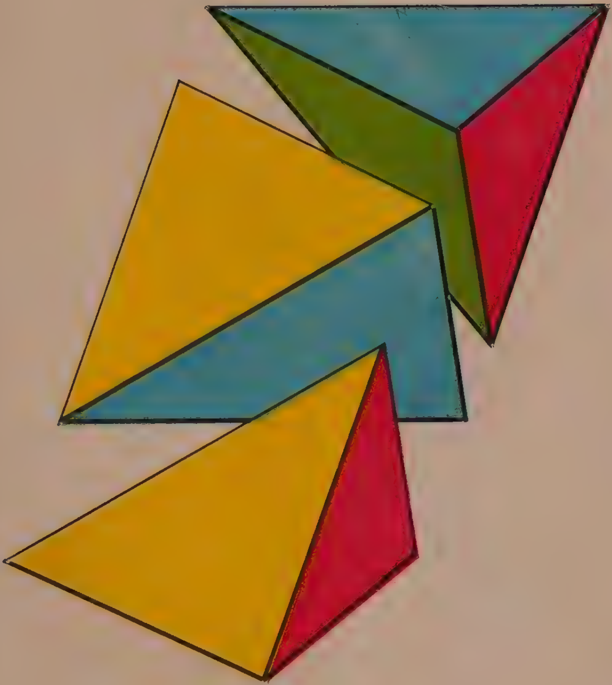


Plate VI.

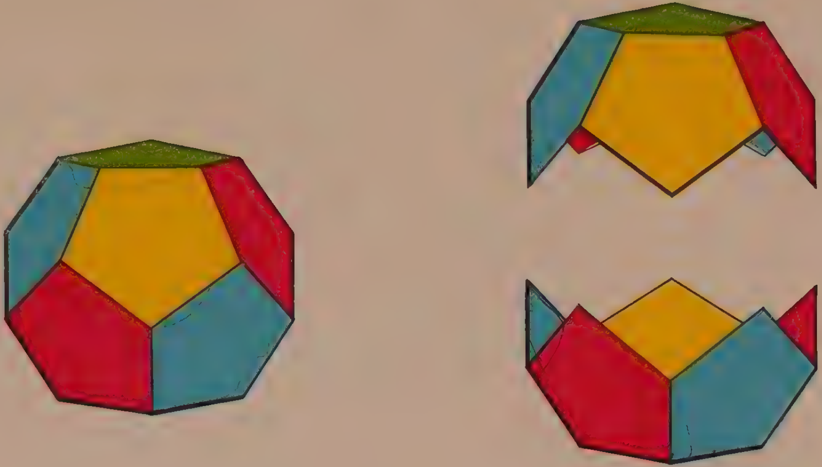
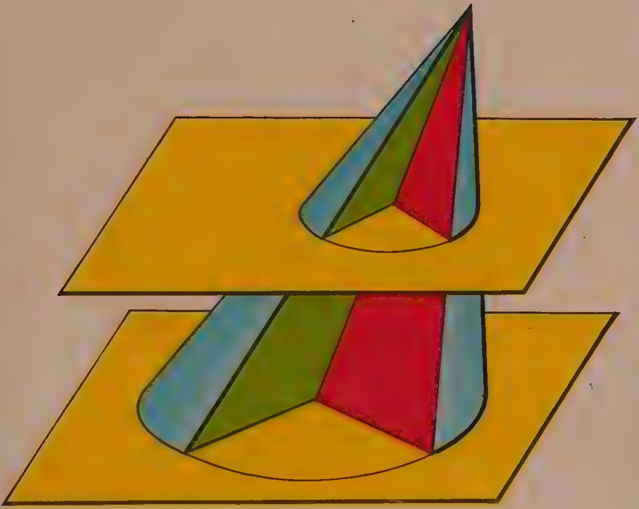


Plate VII,



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ANSWERS

TO

NUMERICAL EXERCISES.

Book I.

4. 24°. **5.** 63° 30′, 26° 30′. **8.** 22° 30′, 157° 30′.
9. 37°. **24.** $A = 112° 30′$, $B = C = 33° 45′$. **88.** 7.

Book II.

12. 28°. **13.** 44° 30′. **14.** 12°. **15.** 54° 30′. **16.** 178°.
17. 112° 30′. **18.** 83°, 89° 30′, 97°, 90° 30′, 74° 30′.
52. $\angle AED = 14° 30′$, $\angle AFB = 10° 30′$.
55. 114° 30′, 89° 30′, 65° 30′, 90° 30′.
67. 97° 30′, 89° 30′, 82° 30′, 90° 30′.

Book III.

1. 112. **2.** 42. **3.** $\frac{25}{27}$. **4.** 63. **5.** BC, $3\frac{1}{5}$, $2\frac{4}{5}$; CA, 4, 3; AB, $4\frac{4}{13}$, $3\frac{9}{13}$. **6.** BC, $11\frac{2}{3}$, $18\frac{2}{3}$; CA, 20, 28; AB, 35, 40.
7. $19\frac{3}{5}$, $25\frac{1}{5}$. **9.** 4 ft. 6 in. **10.** 12. **11.** 15.
12. 37 ft. 1 in. **13.** 47 ft. 6 in. **14.** $\frac{10}{3}\sqrt{3}$.
15. $15\sqrt{2}$ in. **16.** 41. **17.** 58. **18.** 21. **19.** 24.
25. 18. **28.** 48. **29.** 10. **30.** $13\frac{1}{3}$. **31.** $9\sqrt{2}$. **32.** 45.
34. $17\frac{2}{3}$. **37.** 50. **41.** $\sqrt{129}$, $2\sqrt{21}$, $\sqrt{201}$. **42.** $\frac{10}{3}$.
47. 36. **49.** 63. **50.** 4 and 3; $\frac{16}{5}$ and $\frac{9}{5}$. **56.** 24.

57. 17. **58**. 21, 28. **59**. $8\sqrt{3}$. **60**. $BE=4$, $ED=12$. **62**. $6\sqrt{3}$. **67**. 14. **70**. 21. **74**. $\frac{99}{13}$ and $\frac{70}{13}$; 9 and 5.

Book IV.

1. $30\frac{5}{6}$ ft. **2**. 8 ft. 9 in. **3**. 14, 12. **4**. 6 ft. 11 in., 20 ft. 9 in. **5**. 6 sq. ft. 60 sq. in. **6**. $30\sqrt{3}$. **7**. 26 yd. 1 ft. **8**. 2 sq. ft. 48 sq. in. **9**. 243. **10**. 210; $24\frac{12}{17}$, 15, $16\frac{4}{5}$. **11**. 73. **12**. 117. **16**. 2 ft. 10 in. **18**. $\frac{25}{4}\sqrt{3}$. **19**. $3\sqrt{3}$. **21**. 120. **24**. 210. **25**. 18. **26**. $1\frac{1}{2}$ ft. **27**. 6. **28**. $4\sqrt{3}$. **29**. 1260. **33**. 150. **34**. 17. **36**. 624. **37**. 540 sq. in. **38**. 28 ft. **41**. $\frac{13}{3}$. **42**. 30, 16. **45**. $\frac{145}{4}$. **47**. $AD=\frac{15}{2}\sqrt{2}$, $AE=11\sqrt{2}$. **48**. 54. **51**. Area $ABD=39$, area $ACD=45$. **52**. 1010. **53**. 336.

Book V.

32. Area, $\frac{625}{4}\pi$. **33**. Circumference, 34π. **34**. 64 : 121. **36**. 9. **37**. 13. **38**. $\frac{9}{2}\sqrt{2}$. **39**. $\frac{34}{3}\pi$. **40**. $\frac{57}{5}\pi$. **41**. 9.8268. **42**. $\frac{13}{4}\pi$. **43**. 392. **44**. 48π. **45**. 1.2732. **46**. $\frac{64}{9}\pi$. **47**. 6π. **48**. 16π. **49**. 3π, 12π. **50**. 8π, $8\pi\sqrt{2}$. **51**. 9.06. **52**. 416π sq. ft. **53**. 120.99 ft. **54**. 57 in. **60**. 57.295°+. **61**. 2.658+. **62**. 5.64+.

APPENDIX TO PLANE GEOMETRY.

58. $10\sqrt{7}$. **62**. 8. **63**. $\frac{29}{16}$. **91**. 480.

Book VII.

1. 4 : 3. **2**. 2 : 5. **4**. 42. **5**. 1 ft. 9 in. **6**. $34\frac{21}{64}$ cu. in.; $63\frac{3}{8}$ sq. in. **7**. 574. **8**. 1008. **9**. 12, 7. **10**. 1944. **12**. Volume, $50\sqrt{3}$. **14**. Volume, $\frac{243}{2}\sqrt{3}$. **15**. 17. **17**. 2400 sq. in. **18**. 770. **19**. Volume, $48\sqrt{5}$. **20**. 144. **21**. 512, 384. **22**. 1705. **23**. 10, 1. **24**. 36 sq. in. **25**. 12 in. **28**. $\sqrt{273}$, $18\sqrt{237}$, $180\sqrt{3}$. **29**. $\frac{1}{2}\sqrt{118}$,

$3\sqrt{109}$, 15. **30**. $\sqrt{97}$, $12\sqrt{93}$, $72\sqrt{3}$. **31**. $4\sqrt{39}$, $504\sqrt{3}$, $936\sqrt{3}$. **32**. $6\sqrt{3}$, $56\sqrt{26}$, $503\frac{1}{3}$. **33**. $4\sqrt{10}$, $72\sqrt{39}$, $672\sqrt{3}$. **34**. 150. **35**. 320. **36**. 840. **37**. 700, 1568. **38**. $\frac{7}{3}\sqrt{57}$, $640\sqrt{3}$. **39**. $42\sqrt{91}$, $624\sqrt{3}$. **40**. 108, $21\sqrt{39}$. **41**. 240, $\frac{100}{3}\sqrt{119}$. **49**. $4\sqrt{3}$, $\frac{2}{3}\sqrt{2}$. **50**. 15. **51**. 768, 2340. **59**. 438. **63**. 9600 lb. **64**. 50. **69**. $168\sqrt{3}$, $15\sqrt{219}$. **76**. 3456 cu. in. **77**. 6 ft. **78**. 4 ft. 6 in. **79**. $5\sqrt[3]{4}$ in. **80**. 960, 3072. **81**. 128. **82**. 12. **83**. 6. **86**. $36\sqrt{3}$.

Book VIII.

7. $39\frac{2}{3}$. **9**. 86° 24′. **10**. 36. **11**. 44. **12**. $3:2$. **13**. 108°. **14**. 220. **16**. $33\frac{4}{5}$. **17**. $66\frac{1}{2}$. **18**. $36\frac{1}{4}$. **19**. 60. **20**. 153°. **23**. 45°. **24**. $4\frac{4}{5}$. **26**. $4\sqrt{3}$, $\sqrt{3}$. **44**. 30, 8, 20.

Book IX.

1. 288π, 450π, 1296π. **2**. 2420. **3**. 14, 12. **4**. 300π. **5**. 2.7489 +. **6**. 167803.68. **8**. 175π, 224π, 392π. **9**. 143π, 216π, 388π. **10**. 135. **11**. 2800π. **12**. 136π. **13**. 24. **14**. 160π, 536π. **15**. 4, 1159π. **16**. 24, 260π. **17**. 128. **18**. 256. **19**. $\dfrac{32}{3\pi}$ in. **20**. 7238.2464. **22**. 576π. **23**. 416π. **24**. 130π. **25**. 2304π. **26**. 1250π. **27**. 306π. **28**. Volume, 972π. **29**. Area of surface, 225π. **32**. 347.2956. **33**. 56° 15′. **34**. 58π. **37**. 81π, $\frac{243}{2}\pi$. **38**. 8192. **39**. 6 in. **40**. 1625 oz. **41**. 8 in. **42**. 7. **43**. $4\frac{1}{2}$. **48**. $\pi R^2(2-\sqrt{2})$. **49**. $\frac{425}{4}\pi$. **50**. 900π, 4500π. **51**. $36\pi\sqrt{3}$, 54π. **53**. $32\pi\sqrt{3}$. **55**. 720π, 3456π; 960π, 6144π. **56**. $\frac{585}{4}\pi$, $\frac{675}{2}\pi$. **57**. 468π. **58**. $\pi A\sqrt{2}$, $\frac{1}{6}\pi A\sqrt{2A}$.

59. $3\sqrt[3]{9}$ in., $3\sqrt[3]{18}$ in. **60.** $\frac{RT - 2\pi R^3}{2}$. **61.** $\frac{4V}{D}$,
$\frac{8V + \pi D^3}{2D}$. **62.** $\frac{\sqrt{9V^2 + 3\pi H^3 V}}{H}$. **63.** $\frac{S^2\sqrt{\pi^2 L^4 - S^2}}{3\pi^2 L^3}$.
64. $576\pi\sqrt{2}$. **65.** $\frac{S\sqrt{S}}{6\sqrt{\pi}}$. **66.** $\sqrt[3]{36\pi V^2}$.
67. $\frac{\pi(a+b)ab}{\sqrt{a^2+b^2}}$, $\frac{\pi a^2 b^2}{3\sqrt{a^2+b^2}}$. **68.** 2400π.
69. By triangle, πh^2, $\frac{1}{9}\pi h^3$; by inscribed circle, $\frac{4}{9}\pi h^2$, $\frac{4}{81}\pi h^3$.
70. $\frac{4\pi r^2 h^2}{(2r+h)^2}$, $\frac{2\pi r^3 h^3}{(2r+h)^3}$. **71.** $2\pi r^2$, $\frac{1}{2}\pi r^3\sqrt{2}$.
72. $2\pi a^2\sqrt{3}$, $\frac{1}{2}\pi a^3$. **73.** $67.3698+$ lb. **75.** $2\pi a^2\sqrt{3}$, πa^3.
76. $\frac{640}{3}\pi$, $\frac{400}{3}\pi$. **77.** 2100π. **80.** $\frac{2\pi R^2 H}{R+H}$.
81. 1440π. **82.** 487.4716. **83.** $2\pi r^2(1+\sqrt{2})$,
$\frac{1}{3}\pi r^3\sqrt{2}$. **86.** 96π.

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